DOCUMENT RESUME

ED 143 549 SE 023 033

AUTHOR	Clarkson, Donald R., Ed.; And Others
TITLE	Studies in Mathematics, Volume VIII. Concepts of Algebra. Preliminary Edition.
INSTITUTION	Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	61
NOTE	471p.; For related documents, see SE 023 028-041
EDRS PRICE	MF-$0.83 HC-$24.77 Plus Postage.
DESCRIPTORS	*Algebra; *Inservice Education; Instructional Materials; *Number Concepts; *Secondary School Mathematics; *Teaching Guides
IDENTIFIERS	*School Mathematics Study Group

ABSTRACT

This volume is designed to provide information for teachers and prospective teachers who will teach the basic concepts of algebra normally taught in grade 9. Each section of the book contains background information, suggestions for instruction, and problems. Sections in the book include: (1) Numerals and Variables; (2) Open Sentences and English Sentences; (3) The Real Numbers; (4) Properties of Order; and (5) Additive and Multiplicative Inverses. Answers to problems are at the end of the book. (RH)

****
* Documents acquired by ERIC include many informal unpublished *
* materials not available from other sources. ERIC makes every effort *
* to obtain the best copy available. Nevertheless, items of marginal *
* reproducibility are often encountered and this affects the quality *
* of the microfiche and hardcopy reproductions ERIC makes available *
* via the ERIC Document Reproduction Service (EDRS). EDRS is not *
* responsible for the quality of the original document. Reproductions *
* supplied by EDRS are the best that can be made from the original. *
****

ED143549

SCHOOL
MATHEMATICS
STUDY GROUP

STUDIES IN MATHEMATICS
VOLUME VIII
Concepts of Algebra
(preliminary edition)

U S DEPARTMENT OF HEALTH
EDUCATION & WELFARE
NATIONAL INSTITUTE OF
EDUCATION

THIS DOCUMENT HAS BEEN REPRODUCED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT. POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL NATIONAL INSTITUTE OF EDUCATION POSITION OR POLICY

PERMISSION TO REPRODUCE THIS MATERIAL HAS BEEN GRANTED BY

SMSG

TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC) AND USERS OF THE ERIC SYSTEM

SE 023 033

SCHOOL
MATHEMATICS
STUDY GROUP

STUDIES IN MATHEMATICS
VOLUME VIII

Concepts of Algebra

(preliminary edition)

Prepared under the supervision of an Editorial Committee consisting of:

Donald R. Clarkson, North Haven Junior High School,
North Haven, Connecticut

Vincent H. Haag, Franklin and Marshall College

Kenneth Michaels, North Haven High School,
North Haven, Connecticut

Henry O. Pollak, Bell Telephone Laboratories

Charles E. Rickart, Yale University

Written for the SCHOOL MATHEMATICS STUDY GROUP
Under a grant from the NATIONAL SCIENCE FOUNDATION

Copyright 1961 by Yale University
PRINTED IN THE U.S.A.

PHOTOLITHOPRINTED BY CUSHING - MALLOY, INC.
ANN ARBOR, MICHIGAN, UNITED STATES OF AMERICA

CONTENTS

Part — Page

1. NUMERALS AND VARIABLES 1
   SENTENCES AND PROPERTIES OF OPERATIONS 25

2. OPEN SENTENCES AND ENGLISH SENTENCES 59

3. THE REAL NUMBERS 73
   PROPERTIES OF ADDITION 93
   PROPERTIES OF MULTIPLICATION 117

4. PROPERTIES OF ORDER 151
   FIELDS AND MODULO SYSTEMS 172

5. ADDITIVE AND MULTIPLICATIVE INVERSES 177
   FACTORS AND EXPONENTS 206
   POLYNOMIAL AND RATIONAL EXPRESSIONS 237
   TRUTH SETS OF OPEN SENTENCES 284
   SYSTEMS OF EQUATIONS AND INEQUALITIES 322

Answers to Problems . . . . following page 346

Preface

This volume represents a continuation of the SMSG program of providing background material for teachers and prospective teachers - in this case for teachers of the upper grades of the elementary school.

Two other volumes in this series, Number Systems and Intuitive Geometry, have already been prepared for teachers at this level, which is essentially the Level I discussed in the "Recommendations of the Mathematical Association of America for the Training of Teachers of Mathematics*."

This volume and the two mentioned above have been prepared in the belief that teachers at any level should not only thoroughly understand the mathematics they teach, but should also have a good understanding of the basic concepts in the courses which their students will move on to.

Therefore, this volume, which discusses the basic concepts of the algebra normally started in the ninth grade, may also be useful to teachers at the junior high school level as well as at Level I.

It is intended that this volume be revised at some later time. Comments and suggestions for improvements will be welcomed and may be sent to:

School Mathematics Study Group
2502A Yale Station
New Haven, Connecticut

---

*Copies of these recommendations may be obtained from:

Prof. Robert J. Wisner
Committee on the Undergraduate
Program in Mathematics
Michigan State University Oakland
Rochester, Michigan

Part 1

NUMERALS AND VARIABLES

Numerals and Numerical Phrases. Most of your life you have been reading, writing, talking about, and working with *numbers*. You have also used many different names for the same number. Some numbers have one or more common names which are the ones most often used in referring to the numbers. Thus the common name for the number five is "5" and for one gross is "144". Common names for the rational number one-half are "$\frac{1}{2}$" and "0.5". A problem in arithmetic can be regarded as a problem of finding a common name for a number which is given in some other way; for example, 17 times 23 is found by arithmetic to be the number 391.

The *names of numbers*, as distinguished from the *numbers themselves*, are called *numerals*. Two numerals, for example, which represent the same number are the *indicated sum* "4 + 2" and the *indicated product* "2 × 3". The number represented in each case is 6 and we say that "the number 4 + 2 is 6", "the number 2 × 3 is 6", and "the number 4 + 2 is the same as the number 2 × 3". These statements can be written more briefly as "4 + 2 = 6", "2 × 3 = 6", and "4 + 2 = 2 × 3". This use of the equal sign illustrates its general use with numerals: An equal sign standing between two numerals indicates that the *numerals represent the same number*.

We shall need sometimes to enclose a numeral in parentheses in order to make clear that it really is a numeral. Hence it is convenient to regard the symbol obtained by enclosing a numeral for a given number in parentheses as another numeral for the same number. Thus "(4 + 2)" is another numeral for 6 and we might write "(4 + 2) = 6". In order to save writing, the symbol for multiplication "×" is often replaced by a dot "·" . Hence, "2 × 3" can be written as "2·3". Also to save writing, we agree that two numerals placed side-by-side is an indicated product. For example, "2(7 - 4)" is taken to mean the same as "2 × (7 - 4)". Notice, however, that "23" is already established as the common name for the number twenty-three and so cannot be interpreted as the indicated product "2 × 3". A similar exception is "$2\frac{1}{4}$" which means "$2 + \frac{1}{4}$" rather than "$2 \times \frac{1}{4}$". We may, however, write "2(3)" or "(2)(3)" in place of "2 × 3". Similarly, "$2 \times \frac{1}{4}$" might be replaced by "$2 \cdot \frac{1}{4}$" or "$2(\frac{1}{4})$".

Consider the expression "2 × 3 + 7". Is this a numeral? Perhaps you will agree that it is since $2 \times 3 = 6$ and hence,

$$2 \times 3 + 7 = 6 + 7 = 13.$$

On the other hand, someone else might decide that, since $3 + 7 = 10$,

$$2 \times 3 + 7 = 2 \times 10 = 20.$$

Let us examine the expression more carefully. How do we read it? What numerals are involved in it? Obviously "2", "3", and "7" are numerals, but what about "2 × 3" and "3 + 7"? It is true that "2 × 3", as an indicated product, and "3 + 7", as an indicated sum, are both numerals. However, *within the expression* "2 × 3 + 7", if "2 × 3" is an indicated product, then "3 + 7" *cannot* be an indicated sum, or, if "3 + 7" is an indicated sum, then "2 × 3" *cannot* be an indicated product. Therefore, without additional information to decide between these alternatives, the expression "2 × 3 + 7" is really not a numeral since it does not represent a definite number. Another expression in which the same problem arises is "10 - 5 × 2". In order to avoid the confusion in expressions of this kind, we shall agree to give *multiplication preference over addition and subtraction* unless otherwise indicated. In other words, "2 × 3 + 7" will be read with "2 × 3" as an indicated product, so that $2 \times 3 + 7 = 13$. Similarly, "10 - 5 × 2" will be read with "5 × 2" as an indicated product so that $10 - 5 \times 2 = 0$.

Parentheses can also be used to indicate how we intend for an expression to be read. We have only to enclose within parentheses those parts of the expression which are to be taken as a numeral.

Thus, in the case of "$2 \times 3 + 7$", we can write "$(2 \times 3) + 7$" if we want "2×3" to be a numeral and "$2 \times (3 + 7)$" if we want "$3 + 7$" to be a numeral. In other words, the operations indicated within parentheses are taken first. You should always feel free to insert whatever parentheses are needed to remove all doubt as to how an expression is to be read.

Another case in which we need to agree on how an expression should be read is illustrated by the following example:

$$\frac{5(6 - 2)}{13 - 3}.$$

It is understood that the expressions above and below the fraction bar are to be taken as numerals. Therefore the expression is an indicated quotient of the numbers $5(6 - 2)$ and $13 - 3$.

Problems

1. In each of the following, check whether the numerals name the same number.

   (a) $2 + 4 \times 5$ and 22

   (b) $(2 + 4) \times 5$ and 30

   (c) $4 + 3 \times 2$ and $(4 + 3) \times 2$

   (d) $(3 + 2) + 5$ and $3 + (2 + 5)$

   (e) $(4 + \frac{2}{3}) + \frac{1}{3}$ and $4 + (\frac{2}{3} + \frac{1}{3})$

2. Write a common name for each numeral.

   (a) $\frac{1}{2}(5 + 7 \times 3)$

   (b) $4(5) + \frac{9}{3}$

   (c) $\frac{5(6 - 2)}{10 - 3}$

   (d) $\frac{(7 - 2)(3 + 1)}{15}$

   (e) $\frac{6}{8 - 5}$

   (f) $\frac{1 + \frac{1}{2}}{1 - \frac{1}{2}}$

Expressions such as "$4 + 2$", "2×3", "$2(3 + 7)$, "$(1 - \frac{1}{2})(16 + 4) - 5$" are examples of numerical phrases. Each of these is a numeral formed from simpler numerals. A numerical phrase is any numeral given by an expression which involves other numerals along with the signs for operations. It needs to be emphasized that a numerical phrase must actually name a number, that is, it must be a numeral. Therefore, a meaningless expression such as "$(3 \div) \times (2 +) -$" is not a numerical phrase. Even the expression "$2 \times 3 + 7$" is not a numerical phrase without the agreement giving preference to multiplication.

Numerical phrases may be combined to form numerical sentences; i.e., sentences which make statements about numbers. For example,

$$2(3 + 7) = (2 + 3)(4 + 0)$$

is a sentence which states that the number represented by "$2(3 + 7)$" is the same as the number represented by "$(2 + 3)(4 + 0)$". It is read "$2(3 + 7)$ is equal to $(2 + 3)(4 + 0)$", and you can easily verify that it is a true sentence.

Consider the sentence,

$$(3 + 1)(5 - 2) = 10.$$

This sentence asserts that the number $(3 + 1)(5 - 2)$ is 10. Does this bother you? Perhaps you are wondering how the author could have made such a ridiculous mistake in arithmetic, because anyone can see that $(3 + 1)(5 - 2)$ is 12 and not 10! However, "$(3 + 1)(5 - 2) = 10$" is still a perfectly good sentence in spite of the fact that it is false.

The important fact about a sentence involving numerals is that it is either true or false, but not both. Much of the work in algebra is concerned with the problem of deciding whether or not certain sentences involving numerals are true.

Problems

1. Tell which of the following sentences are true and which are false.

    (a) $(3 + 7)4 = 3 + 7(4)$

    (b) $2(5 + \frac{1}{2}) = 2(5) + 2(\frac{1}{2})$

    (c) $\frac{7 + 9}{2} = 7 + \frac{9}{2}$

    (d) $\frac{16}{2} + 4 - 3 = \frac{16}{2} + (4 - 3)$

    (e) $\frac{12}{3(2)} = \frac{12}{3}(2)$

    (f) $3(8 + 2) = 6 \times 5$

    (g) $3 + 7(9 + 2) = (3 + 7)(9 + 2)$

2. Insert parentheses in each of the following expressions so that the resulting sentence is true.

    (a) $3 \times 5 - 2 \times 4 = 7$

    (b) $3 \times 5 - 2 \times 4 = 52$

    (c) $12 \times \frac{1}{2} - \frac{1}{3} \times 9 = 51$

    (d) $12 \times \frac{1}{2} - \frac{1}{3} \times 9 = 3$

    (e) $12 \times \frac{1}{2} - \frac{1}{3} \times 9 = 18$

    (f) $3 + 4 \cdot 6 + 1 = 49$

    (g) $3 + 4 \cdot 6 + 1 = 31$

    (h) $3 + 4 \cdot 6 + 1 = 43$

    (i) $3 + 4 \cdot 6 + 1 = 28$

Some Properties of Addition and Multiplication. Let us summarize some of the properties* of addition. First of all, addition is a binary operation, in the sense that it is always performed on two numbers. When we write $7 + 8 + 3$, we really mean $(7 + 8) + 3$ or $7 + (8 + 3)$. We use parentheses here, as we have in the past, to single out certain groups of numbers to be operated on first. Thus, $(7 + 8) + 3$ implies that we add 7 and 8 and then add 3 to that sum, so that we think "15 + 3".

---

*Property, in the most familiar sense of the word, is something you have. A property of addition is something addition has; i.e., a characteristic of addition. A similar common usage of the word would be "sweetness is a property of sugar".

Similarly, $7 + (8 + 3)$ implies that the sum of 8 and 3 is added to 7, giving $7 + 11$. Let us now go one step further and observe that $15 + 3 = 18$, and $7 + 11 = 18$. We have thus found that

$$(7 + 8) + 3 = 7 + (8 + 3)$$

is a true sentence.

Check whether or not

$$(5 + \frac{3}{2}) + \frac{1}{2} = 5 + (\frac{3}{2} + \frac{1}{2})$$

is a true sentence.

Check similarly

$$(1.2 + 1.8) + 2.6 = 1.2 + (1.8 + 2.6),$$

and

$$(\frac{1}{3} + \frac{1}{2}) + \frac{2}{3} = \frac{1}{3} + (\frac{1}{2} + \frac{2}{3}).$$

It is apparent that these sentences have a common pattern, and they all turned out to be true. We conclude that every sentence having this pattern is true. This is a property of addition of numbers; we hope that you will try to formulate it for yourself. Compare your effort with a statement such as this: If you add a second number to a first number, and then a third number to their sum, the outcome is the same if you add the second number and the third number and then add their sum to the first number.

This property of addition is called the associative property of addition. It is one of the basic facts about the number system--one of the facts on which all of mathematics depends. Incidentally, it is often handy for cutting down the work in adding. In the second example above, for instance, "$\frac{3}{2} + \frac{1}{2}$" is another name for 2, so that "$5 + (\frac{3}{2} + \frac{1}{2})$" produces a simpler addition than "$(5 + \frac{3}{2}) + \frac{1}{2}$". Similarly, in the third example, $1.2 + 1.8 = 3$; which of the two expressions "$(1.2 + 1.8) + 2.6$" and "$1.2 + (1.8 + 2.6)$" leads to a simpler second addition?

Now let us look at the fourth example. Neither "$(\frac{1}{3} + \frac{1}{2}) + \frac{2}{3}$" nor "$\frac{1}{3} + (\frac{1}{2} + \frac{2}{3})$" gives a particularly simple first sum to help us with the second sum. If we could only add $\frac{2}{3}$ to $\frac{1}{3}$ first, this would give 1, and adding $\frac{1}{2}$ to 1 is easy! What we would like is to take the first indicated sum in "$(\frac{1}{3} + \frac{1}{2}) + \frac{2}{3}$", and write it instead as "$(\frac{1}{2} + \frac{1}{3})$", in order to put "$\frac{1}{3}$" next to "$\frac{2}{3}$". To do this, we need to know that

$$\frac{1}{3} + \frac{1}{2} = \frac{1}{2} + \frac{1}{3}$$

is a true sentence.

Is the sentence

$$3 + 5 = 5 + 3$$

true? Perhaps you think: "Walking three miles before lunch and five miles after lunch covers the same distance as walking 5 miles before lunch and 3 miles after lunch."

Now recall the number line.* What do we know about moving from 0 to 5 and then moving three units to the right, and how does this compare with moving from 0 to 3 and then moving 5 units to the right? What does this say about $5 + 3$ and $3 + 5$? Similarly, on the number line, decide whether the following are true sentences:

$$0 + 6 = 6 + 0,$$

$$2\frac{1}{2} + 5 = 5 + 2\frac{1}{2}.$$

This property of addition is probably very familiar to you. It is called the commutative property of addition. If two numbers are added in different orders, the results are the same.

The associative property of addition tells us that an indicated sum may be written with or without parentheses as grouping symbols, as we wish. The commutative property, in turn, tells us that the additions, which are always of two numbers at a time, may be performed in any order. For instance, if we consider

---

*It is assumed that the reader is familiar with the properties of the number line as outlined in Studies in Mathematics, Vol. VI or SMSG Mathematics for Junior High School, Vols. I and II.

$$32 + 16 + 18 + 4,$$

the associative property tells us that we do not have to us parentheses to group this indicated sum, because any way we group it gives the same result. The commutative property tells us that we may choose any other order. For purposes of mental arithmetic, it is easier to choose pairs whose sums are multiples of 10 and consider them first. We may think of "$32 + 16 + 18 + 4$" as "$(32 + 18) + (16 + 4)$", where the indicated sums mean that we first add 32 and 18 (because this gives the "easy" sum 50), then 16 and 4, and finally the two partial sums 50 and 20. In our thinking, we first used the commutative property to interchange the 16 and the 18 in the original indicated sum.

We shall now look at the corresponding properties of multiplication. Consider this sentence,

$$(7 \times 6) \times \frac{1}{3} = 7 \times (6 \times \frac{1}{3}).$$

Check whether or not this is a true sentence; be sure to carry out the multiplications as indicated. We call this property of multiplication the associative property of multiplication. Recall the associative property of addition, and make a similar statement for the associative property of multiplication.

You also know, from arithmetic, that you may perform long multiplication in either order, and you have probably used this to check your work:

$$\begin{array}{r} 256 \\ 63 \\ \hline 768 \\ 1536 \\ \hline 16128 \end{array} \qquad\qquad \begin{array}{r} 63 \\ 256 \\ \hline 378 \\ 315 \\ 126 \\ \hline 16128 \end{array}$$

This is an instance of the commutative property of multiplication: If two numbers are to be multiplied, they may be multiplied, in either order with the same result.

As in the case of addition, the associative and commutative properties of multiplication tell us that we may, in an indicated product, think of the numbers grouped as we choose, and may also rearrange such a product at will. Thus in finding $9 \times 2 \times 3 \times 50$, it is convenient to handle 2×50 first, and then to multiply 9×3, or 27, by 100.

The Distributive Property. Let us assume that one of your students has collected money in his homeroom. On Tuesday, 7 people gave him 15¢ each, and on Wednesday, 3 people gave him 15¢ each. How much money did he collect? He figured,

$$\begin{aligned} 15(7) + 15(3) &= \\ 105 + 45 &= \\ 150. \end{aligned}$$

So he collected $1.50.

But now we shall ask him to keep different records. Since everyone gave him the same amount, it is also possible to keep an account only of the number of people who have paid him, and then to multiply the total number by 15. Then his figuring looks like this:

$$\begin{aligned} 15(7 + 3) &= \\ 15(10) &= \\ 150. \end{aligned}$$

The final result is the same in both methods of keeping accounts; therefore

$$15(7) + 15(3) = 15(7 + 3)$$

is a true sentence. Since the above true sentence means that $15(7) + 15(3)$ and $15(7 + 3)$ are names for the same number, we might also have written

$$15(7 + 3) = 15(7) + 15(3).$$

Half the money John collected is to be used for one gift, and one third of it for another. How much is spent? Again, the computation can be performed in two ways:

$$150(\tfrac{1}{2}) + 150(\tfrac{1}{3}) =$$
$$75 \quad + \quad 50 =$$
$$125.$$

$$150(\tfrac{1}{2} + \tfrac{1}{3}) =$$
$$150(\tfrac{3}{6} + \tfrac{2}{6}) =$$
$$150(\tfrac{5}{6}) =$$
$$125.$$

As before, we have found a true sentence,

$$150(\tfrac{1}{2}) + 150(\tfrac{1}{3}) = 150(\tfrac{1}{2} + \tfrac{1}{3}).$$

Another way of writing the same true sentence would be

$$150(\tfrac{1}{2} + \tfrac{1}{3}) = 150(\tfrac{1}{2}) + 150(\tfrac{1}{3}).$$

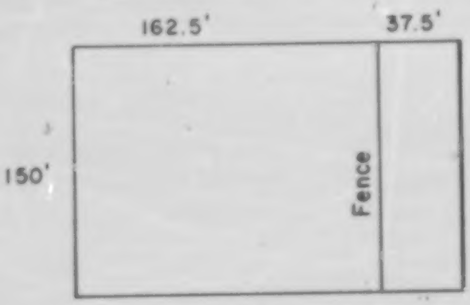
Let us try another example. Mr. Jones owned a city lot, 150 feet deep, with a front of 162.5 feet. Adjacent to his lot, and separated from it by a fence, is another lot with the same depth, but with a front of only 37.5 feet. What are the areas, in square feet, of each of these two lots, and what is their sum? Mr. Jones buys the second lot and removes the fence. Now what is the area of the lot?

The number of square feet in the new lot is

$$150(162.5 + 37.5) =$$
$$150(200) =$$
$$30000.$$

The total area of the two separate lots is

$$150(162.5) + 150(37.5) =$$
$$24375 + 5625 =$$
$$30000.$$



Thus,

$$150(162.5 + 37.5) + 150(162.5) + 150(37.5)$$

is a true sentence.

Let us look more closely at two of our true sentences. We wrote

$$15(7) + 15(3) = 15(7 + 3).$$

$15(7)$ represents one number, which we have chosen to write as an indicated product; so does $15(3)$. Thus $15(7) + 15(3)$ is an indicated sum of two numbers. On the other hand, $7 + 3$ represents a number which we have chosen to write as an indicated sum, and so $15(7 + 3)$ is an indicated product. Thus the sentence

$$15(7) + 15(3) = 15(7 + 3)$$

asserts that the indicated sum and the indicated product are names for the same number. The true sentence

$$150\left(\frac{1}{2} + \frac{1}{3}\right) = 150\left(\frac{1}{2}\right) + 150\left(\frac{1}{3}\right)$$

makes a similar statement.

It appears that we have found another pattern by which true sentences may be formed. Try to formulate in various ways what this pattern is. Compare your result with the following: The product of a number times the sum of two others is the same as the product of the first and second plus the product of the first and third. This property is called the *distributive property for multiplication over addition*, or just, as we shall frequently say, the *distributive property*.

As in the case of the other properties we have studied, the distributive property has much to do with arithmetic, both with devices for mental facility and with the very foundations of the subject. In our first example, the comparison in arithmetic labor between "$15(7) + 12(3)$" and "$15(7 + 3)$" favors the indicated product, because $7 + 3$, or 10, leads to an easy multiplication.

In the next example, however, the comparison between "$150(\frac{1}{2}) + 150(\frac{1}{3})$" and "$150(\frac{1}{2} + \frac{1}{3})$" favors the indicated sum, because it is more work to add the fractions $\frac{1}{2}$ and $\frac{1}{3}$ than it is to add 75 and 50.

More important than these niceties of mental manipulation is the role of the distributive property in much of our arithmetic technique, such as, for example, in long multiplication. How do we perform

$$\begin{array}{r} 62 \\ \underline{\times\ 23} \end{array} \quad ?$$

We write

$$\begin{array}{r} 62 \\ \times\ 23 \\ \hline 186 \\ 124 \\ \hline 1426 \end{array}$$

This really means that we take $62(20 + 3)$ as $62(20) + 62(3)$, or $1240 + 186$. (The "0" at the end of "1240" is understood in our long multiplication form.) Thus the distributive property is the foundation of this standard technique.

Suppose we wish to consider several ways of computing the indicated product

$$(\frac{1}{3} + \frac{1}{4})12.$$

This phrase does not quite fit the pattern of the distributive property as we have discussed it so far. You can probably guess on the basis of your previous experience, that

$$(\frac{1}{3} + \frac{1}{4})12 = (\frac{1}{3})12 + (\frac{1}{4})12$$

is a true sentence. Let us, however, see how the properties as we have discovered them thus far permit us to conclude the truth of this sentence.

First we know that

$$(\frac{1}{3} + \frac{1}{4})12 = 12(\frac{1}{3} + \frac{1}{4})$$

is a true sentence (by what property of multiplication?). Then we may apply the distributive property as we have discovered it thus far to write

$$12(\tfrac{1}{3} + \tfrac{1}{4}) = 12(\tfrac{1}{3}) + 12(\tfrac{1}{4}),$$

and apply the commutative property twice more to write

$$12(\tfrac{1}{3}) + 12(\tfrac{1}{4}) = (\tfrac{1}{3})12 + (\tfrac{1}{4})12.$$

The last step, which would be unnecessary if we were just trying to compute "$(\frac{1}{3} + \frac{1}{4})12$" in a simple fashion, finally leads to the desired sentence, namely,

$$(\tfrac{1}{3} + \tfrac{1}{4})12 = (\tfrac{1}{3})12 + (\tfrac{1}{4})12.$$

This sentence, once again, seems to have a simple form, and in fact suggests an alternate pattern for the distributive property which is obtained from our previous pattern by several applications of the commutative property of multiplication. In your own words, state this alternate pattern. What pattern is suggested by the sentence

$$(\tfrac{1}{3})12 + (\tfrac{1}{4})12 = (\tfrac{1}{3} + \tfrac{1}{4})12?$$

Problems

1. Follow the pattern of any convenient form of the distributive property in completing each of the following as a true sentence:

   (a) $12(3 + 4) = 12(\ \ ) + 12(\ \ )$

   (b) $3(\ \ ) + (7) = 3(5 + 7)$

   (c) $7(\ \ ) + 6(\ \ ) = 13(\ \ )$

   (d) $(3 + 11)2 =$

2. Write a common name for

$$(\frac{1}{2} + \frac{2}{3})11 + (\frac{1}{2} + \frac{2}{3})7.$$

(Hint: Think of $\frac{1}{2} + \frac{1}{3}$ as one numeral, and don't start working until you have thought of a way of doing this exercise which isn't much work.)

3. Write the common names for

(a) $8(\frac{3}{5} + \frac{2}{3}) + (\frac{2}{3} + \frac{3}{5})7$

(b) $7(\frac{1}{2} + \frac{1}{3} + \frac{3}{4}) + 5(\frac{5}{6} + \frac{3}{4})$

(c) $5(\frac{1}{2} + \frac{1}{3} + \frac{1}{5}) + 7(\frac{1}{2} + \frac{1}{3})$

Sets and Subsets. Give a description of the following:

Alabama, Arkansas, Alaska, Arizona?

How would you describe these?

Monday, Tuesday, Wednesday, Thursday, Friday?

Include:

Saturday, Sunday

in the preceding group and then describe all seven. Give a description of the collection of numbers:

1, 2, 3, 4, 5;

of the collection of numbers:

2, 3, 5, 7, 8.

You may wonder if we drifted onto the wrong topic. What do these questions have to do with mathematics? Each of the above collections is an example of a set. Your answers to the questions should have suggested that each set was a particular collection of members or elements with some common characteristic. This characteristic may be only the characteristic of being listed together.

The concept of a set will be one of the simplest of those you will learn in mathematics. A set is merely a collection of elements (usually numbers in our work).

Now we need some symbols to indicate that we are forming or describing sets. If the members of the set can be listed, we may include the members within braces, as for the set of the first five odd numbers:

$$\{1, 3, 5, 7, 9\};$$

or the set of all even numbers between 1 and 9;

$$\{2, 4, 6, 8\};$$

or the set of states whose names begin with C:

{California, Colorado, Connecticut}.

Can you list all the elements of the set of all odd numbers? Or the names of all citizens of the United States? You see that in these cases we may prefer or even be forced to give a description of the set without attempting to list all its elements.

It is convenient to use a capital letter to name a set, such as

$$A = \{1, 3, 5, 7\}$$

or

A is the set of all odd numbers between 0 and 8.

A child learning to count recites the first few elements of a set N which we call the set of counting numbers:*

$$N = \{1, 2, 3, 4, \ldots\}.$$

We write this set with enough elements to show the pattern and then use three dots to mean "and so forth". When we take the set N and include the number 0, we call the new set the set W of whole numbers:

$$W = \{0, 1, 2, 3, 4, \ldots\}.$$

---

*We sometimes refer to counting numbers as "natural numbers".

An interesting question now arises. How shall we describe a set such as the set of all even whole numbers which are greater than 8 and at the same time less than 10? Does this set contain any elements? You may not be inclined to call this a set, because there is no way to list its elements. In mathematics we say that the set which contains no elements is described as the empty, or null, set. We shall use the symbol $\emptyset$ to denote the empty set.

Warning! The set $\{0\}$ is not empty; it contains the element 0. On the other hand, we never write the symbol for the empty set $\emptyset$ with braces.

Perhaps you can think of more examples of the null set, such as the set of all whole numbers between $\frac{4}{3}$ and $\frac{5}{4}$.

Notice that when we talk in terms of sets, we are concerned more with collections of elements than with the individual elements themselves. Certain sets may contain elements which also belong to other sets. For example, let us consider the sets

$$R = \{0, 1, 2, 3, 4\} \text{ and } S = \{0, 2, 4, 6\}.$$

Form the set T of all numbers which belong to both R and S. Thus,

$$T = \{0, 2, 4\}.$$

We see that every element of T is also an element of R. We say that T is a subset of R.

> If every element of a set A belongs to a set B, then A is a subset of B.

Is T a subset of S?

One result is that every set is a subset of itself! Check for yourself that $\{0, \frac{1}{2}, 3, 4\}$ is a subset of itself. We shall also agree that the null set $\emptyset$ is a subset of every set.

Problems

1. Given the following sets:

   P, the set of whole numbers greater than 0 and less than 7;

   Q, the set of counting numbers less than $\frac{13}{2}$;

   R, the set of numbers represented by the symbols on the faces of a die;

   S, the set $\{1, 2, 3, 4, 5, 6\}$.

   (a) List the elements of each of the sets P, Q, R.

   (b) Give a description of set S.

   (c) From the answers to (a) and (b) decide how many possible descriptions a set may have.

2. Find U, the set of all whole numbers from 1 to 4, inclusive. Then find T, the set of squares of all members of U. Now find V, the set of all numbers belonging to both U and T. (Did you include 2 in V? But 2 is not a member of T, so that it cannot belong to both U and T.) Does every member of V belong to U? Is V a subset of U? Is V a subset of T? Is U a subset of T?

3. Returning to Problem 2, let K be the set of all numbers each of which belongs either to U or to T or to both. (Did you include 2 in K? You are right, because 2 belongs to U and hence, belongs to either U or to T. The numbers 1 and 4 belong to both U and T but we include them only once in K.) Is K a subset of U? Is U a subset of K? Is T a subset of K? Is U a subset of U?

4. Consider the four sets

$$\emptyset$$
$$A = \{0\}$$
$$B = \{0, 1\}$$
$$C = \{0, 1, 2\}$$

How many different subsets can be formed from the elements of each of these four sets? Can you tell, without writing out the subsets, the number of subsets in the set

$$D = \{0, 1, 2, 3\}?$$

What is the rule you discovered for doing this?

We shall say that a set is finite if the elements of the set can be counted, with the counting coming to an end, or if the set is the null set. Otherwise, we call it an infinite set. We say that an infinite set has infinitely many elements.

Sometimes a finite set may have so many members that we prefer to abbreviate its listing. For example, we might write the set E of all even numbers between 2 and 50 as

$$E = \{4, 6, 8, \ldots, 48\}.$$

Problems

1. Classify the following sets (finite or infinite):
    - (a) All natural numbers.
    - (b) All squares of all counting numbers.
    - (c) All citizens of the United States.
    - (d) All natural numbers less than one billion.
    - (e) All natural numbers greater than one billion.
2. Given the sets $S = \{0, 5, 7, 9\}$ and $T = \{0, 2, 4, 6, 8, 10\}$.
    - (a) Find K, the set of all numbers belonging to both S and T. Is K a subset of S? Of T? Are S, T, K finite?
    - (b) Find M, the set of all numbers each of which belongs to S or to T or to both. (We never include the same number more than once in a set.) Is M a subset of S? Is T a subset of M? Is M finite?

(c) Find R, the subset of M, which contains all the odd numbers in M. Of which others of our sets is this a subset?

(d) Find A, the subset of R, which contains all the members of M which are multiples of 11. Did you find that A has no members? What do we call this set?

(e) Are sets A and K the same? If not, how do they differ?

(f) From your experience with the last few problems, could you draw the conclusion that subsets of finite sets are also finite?

Variables. Consider, for a moment, a simple exercise in mental arithmetic:

"Take 6, add 2, multiply by 7, and divide by 4."

Following these instructions, you will no doubt think of the succession of numbers

$$6,\ 8,\ 56,\ 14$$

and observe that 14 is the answer to the exercise. Suppose you write down the instructions for each step of the exercise as follows:

$$\begin{gathered} 6 \\ 6 + 2 \\ 7(6 + 2) \\ \frac{7(6 + 2)}{4} \end{gathered}$$

Does this method of writing the exercise give more information or less information? It clearly has the advantage of showing exactly what operations are involved in each step, but it does have the disadvantage of not ending up with an answer to the exercise. On the other hand, the phrase "$\frac{7(6 + 2)}{4}$" is a numeral for the answer 14, a fact which is readily shown by doing the indicated arithmetic.

Here is a situation in which you were asked to record the *form* of the exercise, rather than just the *answer*. It illustrates a point of view which is basic to mathematics. There will be many places in this course where it is the *pattern* or *form* of a problem which is of primary importance rather than the *answer*. *As a matter of fact, we are rarely interested only in the answer to a problem.*

Try the exercise with the following instructions:

"Take 7, multiply by 3, add 3,
multiply by 2, and divide by 12."

What is the phrase which shows all of the operations? Is it a numeral for the same number you obtained mentally?

Let us now do one of these exercises with the added feature that you are permitted to choose at the start any one of the numbers from the set

$$S = \{1, 2, 3, \ldots, 1000\}.$$

The instructions this time are:

"Take any number from S, multiply by 3,
add 12, divide by 3, and subtract 4."

This exercise might be thought of as actually consisting of 1000 different exercises in arithmetic, one for each choice of a starting number from the set S. Consider the exercise obtained by starting with the number 17. The arithmetic method and the pattern method lead to the following steps:

Arithmetic	Pattern
17	17
51	$3(17)$
63	$3(17) + 12$
21	$\frac{3(17) + 12}{3}$
17	$\frac{3(17) + 12}{3} - 4$

Notice that, by the distributive property for numbers and since $12 = 3(4)$,

$$3(17) + 12 = 3(17) + 3(4) = 3(17 + 4),$$

so that

$$\frac{3(17) + 12}{3} = \frac{3(17 + 4)}{3} = 17 + 4.$$

Therefore

$$\frac{3(17) + 12}{3} - 4 = 17 + 4 - 4 = 17.$$

In other words, the final phrase obtained in the "pattern" is a numeral for 17. Try some more choices from the set S. Will you always end up with the number you chose at the start? One method of answering this question would be to work out each of the 1000 different exercises! Perhaps you have already guessed, from working the pattern for several cases, that it may not be necessary to do all of the 1000 exercises to answer the above question.

Let us examine the pattern of the exercise more closely. Observe first that the pattern really does not depend on the number chosen from the set S. In fact, if we refer to the number chosen by the word "number", the steps in the exercise become:

$$\begin{gathered} \text{number} \\ 3(\text{number}) \\ 3(\text{number}) + 12 \\ \frac{3(\text{number}) + 12}{3} \\ \frac{3(\text{number}) + 12}{3} - 4 \end{gathered}$$

In order to save writing, denote the chosen number by "n". Then the steps become:

$$\begin{gathered} n \\ 3n \\ 3n + 12 \\ \frac{3n + 12}{3} \\ \frac{3n + 12}{3} - 4 \end{gathered}$$

Note that "n" is used here as a numeral for the chosen number and that the phrase in each of the other steps is also a numeral. (Thus, if n happens to be 17, then the indicated product "3n" is a numeral for 51.) In particular, the phrase "$\frac{3n + 12}{3} - 4$" is a numeral for the "answer" to the exercise. Moreover, by the distributive property for numbers,

$$3n + 12 = 3n + 3(4) = 3(n + 4).$$

Hence,

$$\frac{3n + 12}{3} = \frac{3(n + 4)}{3} = n + 4$$

Therefore,

$$\frac{3n + 12}{3} - 4 = n + 4 - 4 = n.$$

Since "n" can represent any particular element of the set S, we conclude that the end result in this exercise is indeed always the same as the number chosen at the start.

The above discussion illustrates the great power of methods based on pattern or form rather than on simple arithmetic. The method, in a sense, enables us to replace 1000 different arithmetic problems by a single problem!

Would the discussion of the exercise be changed in any essential way if we had decided to denote the chosen number from S by some letter other than n, say m or x?

A letter used, as "n" was used in the above exercise, to denote one of a given set of numbers, is called a variable. In a given computation involving a variable, the variable is a numeral which represents a definite though unspecified number from a given set of admissible numbers. The admissible numbers for the variable "n" in the above exercise are the whole numbers from 1 to 1000. A number which a given variable can represent is called a value of the variable. The set of values of a variable is sometimes called its domain. The domain of the variable "n" in the above exercise is the set S = {1, 2, 3, ..., 1000}. Unless the domain of a variable is specifically stated, we shall assume it to be the set of all numbers. For the time being, we are considering only the numbers of arithmetic.

Problems

1. If the sum of a certain number t and 3 is doubled, which of the following would be a correct form:

$$2t + 3 \quad \text{or} \quad 2(t + 3)?$$

2. If 5 is added to twice a certain number n and the sum is divided by 3, which is the correct form:

$$\frac{2n + 5}{3} \quad \text{or} \quad \frac{2n}{3} + 5?$$

3. If one-fourth of a certain number x is added to one third of four times the same number, which is the correct form:

$$\frac{1}{3}(4x) + \frac{1}{4}(x) \quad \text{or} \quad \frac{4}{3}(x) + \frac{1}{4}(x)?$$

4. If the number of gallons of milk purchased is y, which is the correct form for the number of quart bottles that will contain it:

$$4y \quad \text{or} \quad \frac{y}{4}?$$

5. If a is the number of feet in the length of a certain rectangle and b is the number of feet in the width of the same rectangle, is either form the correct form for the perimeter:

$$a + b \quad \text{or} \quad ab?$$

6. If a is 2, b is 3, c is $\frac{3}{4}$, m is 1 and n is 0, then what is the value of (that is, what number is represented by):

(a) $6b + ac$

(b) $(a + b)(a + m)$

(c) $6(b + ac)$

(d) $\frac{2b + c}{b}$

(e) $nc + ac$

(f) $n(c + ac)$

(g) $\frac{2a + 3b}{m}$

(h) $m(b - 4c)$

(i) $\frac{5(3a + 4b)}{2(a + b)}$

(j) $a + 2(b + m)$

7. If a is 3, b is 2, and c is 4, then what is the value of:

(a) $\frac{(3a + 4b) - 2c}{3}$?

(b) $\frac{(6a - 4b) + 5c}{5}$?

(c) $\frac{\left(\frac{7a}{2} + \frac{3b}{2}\right) - \frac{5c}{2}}{2}$?

(d) $\frac{(1.5a + 3.7b) - 2.1c}{7}$?

8. Summarize the new ideas, including definitions, which have been introduced so far.

SENTENCES AND PROPERTIES OF OPERATIONS

Sentences, *True* *and* *False*. Is the sentence, "The cover of this book is blue." true or false? Obviously, it is true. Let us consider another sentence. "The world is flat." is obviously false. If one were to say "_______ is president of the United States", would this sentence be true or false? Actually it is neither. The sentence may be made true by inserting the proper name in the blank, but it could just as easily be made false. We say that it is "open", i.e., it is neither true nor false until we complete it, but once completed it then will be either true *or* false.

The choice we make to complete the sentence will determine whether it is to be true or false. It should be noted, however, that it is permissable (from the standpoint of logic) to make a false statement.

When we make assertions about numbers we also write *sentences*, such as

$$(3 + 1)(5 + 2) = 10$$

Remember that a sentence is either true or false, but not both. This particular sentence is false.

Some sentences, such as the one above, involve the verb "$=$", meaning "is" or "is equal to". There are other verb forms that we shall use in mathematical sentences. For example, the symbol "$\neq$" will mean "is not" or "is not equal to". Then

$$8 + 4 \neq 28 \cdot 2$$

is a true sentence, and

$$8 + 4 \neq \frac{24}{2}$$

is a false sentence.

Problems

Which of the following sentences are true? Which are false?

1. $4 + 8 = 10 + 5$
2. $5 + 7 \neq 6 + 6$
3. $\frac{1}{2} + \frac{5}{8} = 1 + \frac{1}{8}$
4. $\frac{85}{1} \neq 85$
5. $7(6 \times 3) = (7 \times 6) \times 3$
6. $8(\frac{1}{2} - \frac{1}{4}) = 8(\frac{1}{2}) - 8(\frac{1}{4})$
7. $13 \times 0 = 13$
8. $\frac{2}{3}(7) \neq 2(\frac{7}{3})$
9. $8(\frac{3}{5}) = \frac{24}{40}$

Open Sentences. We have no trouble deciding whether or not a numerical sentence is true, because such a sentence involves specific numbers. Consider the sentence

$$x + 3 = 7.$$

Is this sentence true? You will protest that you don't know what number "x" represents; without this information you cannot decide. In the same way you cannot decide whether the sentence, "He is a doctor," is true until "he" is identified. In this sense, the variable "x" is used in much the same way as a pronoun in ordinary language.

Consider the sentence

$$3n + 12 = 3(n + 4).$$

We cannot decide whether this sentence is true on the basis of the sentence alone, but here we have a different situation. We could decide if we knew what number "n" represents. But in this case we can decide without knowing the value of n. We can recall a general property of numbers to show that this sentence is true *no matter what number "n" represents*. (What did we call this general property of numbers?)

We say that sentences such as

$$x + 3 = 7$$

and

$$3n + 12 = 3(n + 4),$$

which contain variables, are open sentences. The word "open" is suggested by the fact that we do not know whether they are true without more information. An open sentence is a sentence involving one or more variables, and the question of whether it is true is left open until we have enough additional information to decide. In the same way, a phrase involving one or more variables is called an open phrase.

Problems

In each of the following open sentences, determine whether the sentence is true if the variables have the suggested values:

1. $7 + x = 12$; let x be 5.
2. $7 + x \neq 12$; let x be 5.
3. $\frac{5x + 1}{7} \neq 3$; let x be 3; let x be 4.
4. $2y + 5x = 23$; let x be 4 and y be 3; let x be 3 and y be 4.
5. $2a - 5 \neq (2a + 4) - b$; let a be 9 and b be 9; let a be 3 and b be 9.

If we are given an open sentence, and the domain of the variable is specified, how shall we determine the values, if any, of the variable that will make it a true sentence? We could guess various numbers until we hit on a "truth" number, but after the first guess, a bit of thinking could guide us.

Let us experiment with the open sentence "$2x - 11 = 6$". As a first guess, try a number x large enough so that $2x$ is greater than 11; let x be 9. Then

$$\begin{aligned} 2x - 11 &= 2(9) - 11 \\ &= 7. \end{aligned}$$

Thus, the numeral on the left represents 7, which is different from 6. Apparently 9 was too large; so we try 8 for x. Then

$$\begin{aligned} 2x - 11 &= 2(8) - 11 \\ &= 5. \end{aligned}$$

Here the numeral on the left represents 5, which is also different from 6. Since 8 was too small, we try a number between 8 and 9, say $8\frac{1}{2}$. Then

$$\begin{aligned} 2x - 11 &= 2(8\tfrac{1}{2}) - 11 \\ &= 6. \end{aligned}$$

"$6 = 6$" is a true sentence; so we find that the open sentence "$2x - 11 = 6$" is true if x is $8\frac{1}{2}$. Do you think there are other values of x making it true? Do you think every open sentence has a value of the variable which makes it true? Which makes it false?

Problems

Determine what numbers, if any, will make each of the following open sentences true:

1. $12 + y = 8$
2. $4x - 3x = 14$
3. $s + 3 = s + 2$
4. $t + 2t \neq 27 + 3t$
5. $t + 3 = 3 + t$
6. $(x + 1)2 \neq 2x + 2$

If a variable occurs in an open sentence in the form "$a \cdot a$" meaning "a multiplied by a", it is convenient to write "$a \cdot a$" as "a^2", read "a squared".

Problems

1. Try to find values of the variables which make the following open sentences true:

   (a) $x^2 = 9$ (d) $(x - 1)^2 = 4$

   (b) $4 - x^2 = 0$ (e) $4 + x^2 = 0$

   (c) $x^2 = x$ (f) $x^2 + 7 = 7$

2. What is a value of x for which

   $$x^2 = \frac{9}{16}$$

   is a true sentence?

3. A number of interest to us later is a value of x for which "$x^2 = 2$" is a true sentence. We call this number the square root of 2, and write it $\sqrt{2}$. Later you will find that $\sqrt{2}$ is the coordinate of a point on the number line. Approximately where on the number line would it lie?

Truth Sets of Open Sentences. Let the domain of the variable in the sentence

$$3 + x = 7$$

be the set of all numbers of arithmetic. If we specify that x has a particular value, then the resulting sentence is true or is false. For instance,

If x is	the sentence	is
0	$3 + 0 = 7$	false
1	$3 + 1 = 7$	false
$\frac{1}{2}$	$3 + \frac{1}{2} = 7$	false
2	$3 + 2 = 7$	false
4	$3 + 4 = 7$	true
6	$3 + 6 = 7$	false

In this way the sentence "$3 + x = 7$" can be thought of as a *sorter*: it sorts the domain of the variable into two subsets. Just as you might sort a deck of cards into two subsets, black and red, the domain of the variable is sorted into a subset of all those numbers which make the sentence *true* and another subset of all those numbers which make the sentence *false*. Here we see that 4 belongs to the first subset, while $0, 1, \frac{1}{2}, 2, 6$ belong to the second subset.

The *truth set of an open sentence in one variable is the set of all those numbers from the domain of the variable which make the sentence true*. If we do not specify otherwise we shall continue to assume that the domain of the variable is the set of all numbers of arithmetic. (Recall that the numbers of arithmetic consist of 0 and all numbers which are coordinates of points to the right of 0 on the number line.)

Problems

1. With each of the following open sentences is given a set which contains all the numbers belonging to its truth set, with possibly some more. You are to find the truth set.

   (a) $3(x + 5) = 17$; $\{0, \frac{1}{3}, \frac{2}{3}, 1\}$.

   (b) $x^2 - (4x - 3) = 0$; $\{1, 2, 3, 4\}$.

   (c) $x^2 - \frac{7}{6}x + \frac{1}{6} = 0$; $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}\}$.

   (d) $x + \frac{1}{x} = 2$; $\{1, 2, 3\}$.

   (e) $x(x + 1) = 3x$; $\{0, 1, 2\}$.

   (f) $\frac{5x + 1}{7} = 3$; $\{0, 2, 4\}$.

(g) $x + 1 = 5x - 1$; $\{1, \frac{1}{2}, 2\}$.

(h) $x + 2 = x + 7$; $\{0, 2, 3\}$.

2. Write an open sentence whose truth set is the null set $\emptyset$.

Many formulas used in science and business are in the forms of open sentences in several variables. For example, the formula

$$V = \frac{1}{3}Bh$$

is used to find the volume of a cone. The variable h represents the number of units in the height of the cone; B represents the number of square units in the base; V represents the number of cubit units in the volume. When values are specified for all but one of the variables in such a formula, the resulting open sentence contains one remaining variable. Then the truth set of this sentence gives information about the number represented by this variable.

Continuing the example, let us consider a particular cone whose volume is 66 cubic feet and the area of whose base is 33 square feet. From this information we determine that V is 66 and B is 33, and we write the corresponding open sentence in one variable h,

$$66 = \frac{1}{3}(33)h.$$

The truth set of this sentence is $\{6\}$.

Applying this information to the cone, we find that the height of the cone is 6 feet.

Problems

1. The formula used to change a temperature F measured in degrees Fahrenheit to the corresponding temperature C in degrees Centigrade is

$$C = \frac{5}{9}(F - 32).$$

Find the value of C when F is 86.

2. A formula used in physics to relate pressure and volume of a given amount of a gas at constant temperature is

$$pv = PV,$$

where V is the number of cubic units of volume at P units of pressure and v is the number of cubic units of volume at p units of pressure. Find the value of V when v is 600, P is 75, and p is 15.

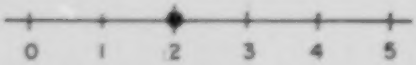
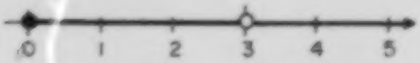
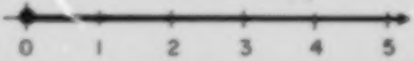
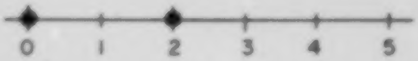
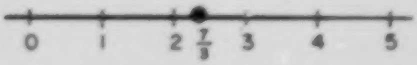
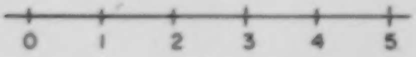
3. The formula for the area of a trapezoid is

$$A = \frac{1}{2}(B + b)h,$$

where A is the number of square units in the area, B is the number of units in the one base, b is the number of units in the other base, and h is the number of units in the height. Find the value of B when A is 20, b is 4, and h is 4.

Graphs of Truth Sets. The graph of a set S of numbers, it will be recalled, is the set of all points on the number line corresponding to the numbers of S, and only those points.

Thus, the graph of the truth set of an open sentence containing one variable is the set of all points to the right of 0 on the number line whose coordinates are the values of the variable which make the open sentence true. Let us draw the graphs of a few open sentences.

	Sentence	Truth Set	Graph
(a)	$x = 2$	$\{2\}$	
(b)	$x \neq 3$	All numbers of arithmetic except 3	
(c)	$3 + x = (7 + x) - 4$	All numbers of arithmetic	
(d)	$y(y + 1) = 3y$	$\{0, 2\}$	
(e)	$3y = 7$	$\{\frac{7}{3}\}$	
(f)	$2x + 1 = 2(x + 1)$	$\emptyset$	 (Graph contains no points)

You will notice in (b) that we indicate that a point is included in the graph if it is marked with a heavy dot, but not included if it is circled. The heavy lines indicate all the points that are covered. The arrow at the right end of the number line in (b) and (c) indicates that all of the points to the right are on the graph.

Problems

State the truth set of each open sentence and draw its graph:

1. $x + 7 = 10$
2. $2x = x + 3$
3. $x + x \neq 2x$
4. $x + 3 = 3 + x$
5. $(x)(0) = x$
6. $2x + 3 = 8$
7. $y \cdot (1) \neq y$
8. $x^2 = 2x$

Sentences Involving Inequalities. If we consider any two different numbers, then one is less than the other. Is this always true? This suggests another verb form that we shall use in numerical sentences. We use the symbol "<" to mean "is less than" and ">" to mean "is greater than".

To avoid confusing these symbols, remember that in a true sentence, such as

$$8 < 12$$

or

$$12 > 8,$$

the point of the symbol (the small end) is directed toward the smaller of the two numbers.

Find the two points on the number line which correspond to 8 and 12. Which point is to the left? Will the lesser of two numbers always correspond to the point on the left of the other? Verify your answer by locating on the number line points corresponding to several pairs of numbers, such as $\frac{5}{2}$ and 2.2; $\frac{8}{6}$ and $\frac{6}{5}$.

Just as "$\neq$" means "is not equal to", "$\ngtr$" means "is not greater than". What does "$\nless$" mean?

Problems

Which of the following sentences are true? Which are false? Use the number line to help you decide.

1. $4 + 3 < 3 + 4$
2. $5(2 + 3) > 5(2) + 3$
3. $\frac{1}{2} + \frac{1}{3} \nless 1$
4. $5 + 0 \nless 5$
5. $2 > 2 \times 0$
6. $0.5 + 1.1 = 0.7 + 0.9$
7. $5.2 - 3.9 < 4.6$
8. $2 + 1.3 > 3.3$
9. $2 + 1.3 \ngtr 3.3$
10. $4 + (3 + 2) < (4 + 3) + 2$
11. $\frac{2}{3}(8 + 4) < \frac{2}{3} \times 8 + \frac{2}{3} \times 4$
12. $5 + (\frac{2}{5} + \frac{3}{5}) \neq (4 - 1)2$

Open Sentences Involving Inequalities. What is the truth set of the open sentence

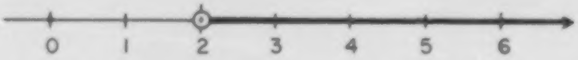
$$x + 2 > 4?$$

We can answer this question as follows: We know that the truth set of

$$x + 2 = 4$$

is $\{2\}$. When x is a number greater than 2, then $x + 2$ is a number greater than 4. When x is a number less than 2, then $x + 2$ is a number less than 4. Thus, every number greater than 2 makes the sentence true, and every other number makes it false. That is, the truth set of the sentence "$x + 2 > 4$" is the set of all numbers greater than 2.

The graph of this truth set is the set of all points on the number line whose coordinates are greater than 2. This is the set of all points which lie to the right of the point with coordinate 2:



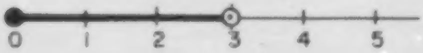
As another example, consider the graph of the truth set of

$$1 + x < 4:$$

Truth Set

All numbers of arithmetic from 0 to 3, including 0, not including 3.

Graph



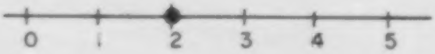
It is customary to call a simple sentence involving "=" an equation and a sentence involving "<" or ">" an inequality.

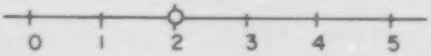
Problems

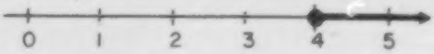
1. Draw the graphs of the truth sets of the following open sentences:

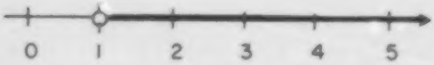
(a) $x \neq 2$
(b) $x > 2$
(c) $3 + y \neq 4$
(d) $3 + y > 4$
(e) $3 + y < 4$
(f) $m < 3$
(g) $m > 3$

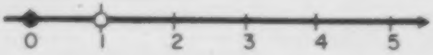
2. Below are some graphs. For each graph, find an open sentence whose truth set is the set whose graph is given.

(a)


(b)


(c)


(d)


(e)


3. If the domain of the variable of each open sentence below is the set $\{0, 1, 2, 3, 4, 5\}$, find the truth set of each, and draw its graph.

(a) $4 + x = 7$
(b) $4x + 3 = 6$
(c) $2x > 5$
(d) $x + 3 < 6$
(e) $2x + 6 = 2(x + 3)$

Sentences With More Than One Clause. All the sentences discussed so far have been simple--that is, they contained only one verb form. Let us consider a sentence such as

$$4 + 1 = 5 \text{ and } 6 + 2 = 7.$$

Your first impression may be that we have written two sentences. But if you read the sentence from left to right, it will be one compound sentence with the connective and between two clauses. So in mathematics, as well as in English, we encounter sentences (declarative sentences) which are compounded out of simple sentences.

Recall that a numerical sentence is either true or false. The compound sentence

$$4 + 1 = 5 \text{ and } 6 + 2 = 7$$

is certainly false, because the word and means "both", and here the second of the two clauses is false. The compound sentence

$$3 \ngtr 1 + 2 \text{ and } 4 + 7 > 10$$

is true, because both clauses are true sentences.

In general, a compound sentence with the connective and is true if all its clauses are true sentences; otherwise, it is false.

Problems

Which of the following sentences are true?

1. $4 = 5 - 1$ and $5 = 3 + 2$
2. $5 = \frac{11}{2} - \frac{1}{2}$ and $6 < \frac{2}{3} \times 9$
3. $3 > 3 + 2$ and $4 + 7 < 11$
4. $3 + 2 > 9 \times \frac{1}{3}$ and $4 \times \frac{3}{2} \ngtr 5$
5. $3.2 + 9.4 \nless 12.6$ and $\frac{7}{8} < \frac{11}{12}$
6. $3.25 + 0.3 \nless 6.25$

Consider next the sentence

$$4 + 1 = 5 \text{ or } 6 + 2 = 7.$$

This is another type of compound sentence, this time with the connective or. Here we must be very careful. Possibly we can get a hint from English sentences. If we say, "The Yankees or the Indians will win the pennant", we mean that exactly one of the two will win; certainly, they cannot both win. But when we say, "My package or your package will arrive within a week", it is possible that both packages may arrive; here we mean that one or more of the packages will arrive, including the possibility that both may arrive. The second of these interpretations of "or" is the one which turns out to be the better suited for our work in mathematics.

Thus, we agree that a compound sentence with the connective or is true if one or more of its clauses is a true sentence; otherwise, it is false.

We classify

$$4 + 1 = 5 \text{ or } 6 + 2 = 7$$

as a true compound sentence, because its first clause is a true sentence; we also classify

$$5 < 4 + 3 \text{ or } 2 + 1 \neq 4$$

as a true compound sentence, because one or more of its clauses is true (here both are true).

Is the sentence

$$3 \neq 2 + 1 \text{ or } 2 > 4 + 1$$

false? Why?

Problems

Which of the following sentences are true?

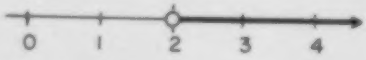
1. $3 = 5 - 1$ or $5 = 3 + 2$
2. $7 = \frac{11}{2} + \frac{3}{2}$ or $2 = \frac{11}{2} - \frac{3}{2}$
3. $4 > 3 + 2$ or $6 < 3 + 1$
4. $2 + 3 > 9 \times \frac{1}{3}$ and $4 \times \frac{3}{2} \ngtr 6$
5. $6.5 + 2.3 \ngtr 8.8$ or $\frac{3}{5} < \frac{7}{15}$
6. $5 + 4 < 9$ or $\frac{3}{4} < \frac{9}{12}$.

Graphs of Truth Sets of Compound Open Sentences. Our problems in graphing have so far involved only simple sentences. Graphs of compound open sentences require special handling. Let us consider the open sentence

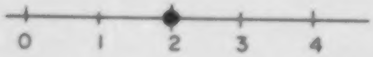
$$x > 2 \text{ or } x = 2.$$

The clauses of this sentence and the corresponding graphs of their truth sets are:

$x > 2$

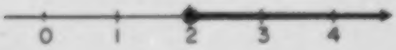


$x = 2$



If a number belongs to the truth set of the sentence "$x > 2$" or to the truth set of the sentence "$x = 2$", it is a number belonging to the truth set of the compound sentence "$x > 2$ or $x = 2$". Therefore, every number greater than or equal to 2 belongs to the truth set. On the other hand, any number less than 2 makes both clauses of the compound sentence false and so fails to belong to its truth set. The graph of the truth set is then

$x > 2$ or $x = 2$



We abbreviate the sentence "$x > 2$ or $x = 2$" to "$x \geq 2$", read "x is greater than or equal to 2". Give a corresponding meaning for "$\leq$".

Let us make a precise statement of the principle involved:

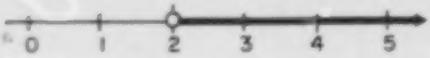
The graph of the truth set of a compound sentence with connective or consists of the set of all points which belong to either one of the graphs of the two clauses of the compound sentence.

Finally, we consider the problem of finding the graph of an open sentence such as

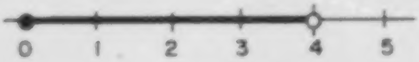
$$x > 2 \text{ and } x < 4.$$

Again we begin with the two clauses and the graphs of their truth sets:

$x > 2$

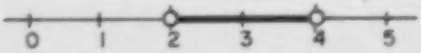


$x < 4$



Then it follows (using an argument similar to that above) that the graph of the truth set of the compound sentence is

$x > 2$ and $x < 4$



Sometimes we write "$x > 2$ and $x < 4$" as "$2 < x < 4$".

We see that the graph of the truth set of a compound sentence with connective and consists of all points which are common to the graphs of the truth sets of the two clauses of the compound sentence.

It has required many words, carefully chosen, to describe the various connections between sentences, truth sets and graphs. We consistently referred to the graph of the truth set of an open sentence. In the future, let us shorten this phrase to the graph of a sentence. It will be a simpler description, and no confusion will result if we recall what is really meant by the description.

For the same reasons we shall find it convenient to speak of the point 3, or the point $\frac{1}{2}$, when we mean the point with coordinate 3 or the point with coordinate $\frac{1}{2}$. Points and numbers are distinct entities to be sure, but they correspond exactly on the number line. Whenever there is any possibility of confusion we shall remember to give the complete descriptions.

Problems

Construct the graphs of the following open sentences:

1. $x = 2$ or $x = 3$
2. $x = 2$ and $x = 3$
3. $x > 5$ or $x = 5$
4. $x > 2$ and $x < \frac{11}{2}$
5. $x > 5$ and $x = 5$
6. $x > 3$ or $x = 3$

Summary of Open Sentences. We have examined some sentences and have seen that each one can be classified as either true or false, but not both. We have established a set of symbols to indicate relations between numbers:

"$=$" means "is" or "is equal to"
"$\neq$" means "is not" or "is not equal to"
"$<$" means "is less than"
"$>$" means "is greater than"
"$\leq$" means "is less than or is equal to"
"$\geq$" means "is greater than or is equal to".

We have discussed compound sentences which have two clauses. If the clauses are connected by the word or, the sentence is true if at least one clause is true; otherwise it is false. If the clauses are connected by and, the sentence is true if both clauses are true; otherwise it is false.

An open sentence is a sentence containing one or more variables.

The truth set of an open sentence containing one variable is the set of all those numbers which make the sentence true. The open sentence acts as a sorter, to sort the domain of the variable into two subsets: a subset of numbers which make the sentence true, and a subset which make the sentence false.

The graph of a sentence is the graph of the truth set of the sentence.

Problems

State the truth set of each of the following open sentences and construct its graph. Some examples of how you might state the truth sets are:

Open Sentence	Truth Set
$x + 3 = 5$	$\{2\}$.
$2x \neq x + 3$	The set of all numbers of arithmetic except 3.
$x + 1 < 5$	The set of all numbers of arithmetic less than 4.
$2x \geq 9$	The set of numbers consisting of $4\frac{1}{2}$ and all numbers greater than $4\frac{1}{2}$.

1. $z + 8 = 14$
2. $2 + v < 15$
3. $6 > 1 + 3$ and $5 + t = 4$
4. $6 > 1 + 3$ or $2 + t = 1$
5. $x + 2 = 3$ or $x + 4 = 6$
6. $\frac{x}{2} > 3$
7. $t + 4 = 5$ or $t + 5 \neq 5$
8. $3a \neq a + 5$
9. $9 + t < 12$ or $5 + 1 \neq 6$
10. $5x + 3 < 19$
11. $(x - 1)^2 = 4$
12. $3x = (8 + x) - 2$
13. $t + 6 \leq 7$ and $t + 6 \geq 7$

Identity Elements. For what number n is it true that

$$n + 0 = n?$$

Here we have an interesting property which we shall call the addition property of zero. We can state this property in words: "The sum of any number and 0 is equal to the number."

We can state this property in the language of algebra as follows:

For every number a,

$$a + 0 = a.$$

Since adding 0 to any number gives us identically the same number, 0 is often called the identity element for addition.

Is there an identity element for multiplication? Consider the truth sets of the following open sentences:

$$3x = 3$$
$$\frac{2}{3}n = \frac{2}{3}$$
$$.7 = .7y$$
$$n(5) = 5.$$

You have surely found that the truth set of each of these is $\{1\}$. Thus,

$$n(1) = n$$

seems to be a true sentence for all numbers. How could you state in words this property, which we shall call the multiplication property of one?

We can also state this in the language of algebra:

For every number a,

$$a(1) = a.$$

We can see that the identity element for multiplication is 1.

There is another property of zero which will be evident if you answer the following questions:

1. What is the result when any number of arithmetic is multiplied by 0?

2. If the product of two numbers is 0, and one of the numbers is 0, what can you tell about the other number?

The property that becomes apparent is called the multiplication property of zero, and can be stated as follows:

For every number a,

$$a(0) = 0.$$

These properties of zero and 1 are very useful. For instance, we use the multiplication property of 1 in arithmetic in working with rational numbers. Suppose we wish to find a numeral for $\frac{5}{6}$ in the form of a fraction with 18 as its denominator. Of the many names for 1, such as $\frac{2}{2}$, $\frac{3}{3}$, $\frac{5}{5}$, ..., we choose "$\frac{3}{3}$" because 3 is the number which multiplied by 6 gives 18. We then have

$$\begin{aligned}\frac{5}{6} &= \frac{5}{6}(1)\\ &= \frac{5}{6}\left(\frac{3}{3}\right)\\ &= \frac{5(3)}{6(3)}\\ &= \frac{15}{18}.\end{aligned}$$

Suppose we now wish to add $\frac{7}{9}$ to $\frac{5}{6}$. To do this, we desire other names for $\frac{7}{9}$ and $\frac{5}{6}$, names which are fractions with the same denominator. What denominator should we choose? It must be a multiple of both 6 and 9, but it cannot be 0. Thus, 36, or 18, or 54, or many others, are possible choices. For simplicity we pick the smallest, which is 18. (This is called the least common multiple of 6 and 9.) In order, now, to add $\frac{7}{9}$ to $\frac{5}{6}$, we already know that

$$\frac{5}{6} = \frac{15}{18}.$$

Similarly,

$$\begin{aligned}\frac{7}{9} &= \frac{7}{9}(1)\\ &= \frac{7}{9}\left(\frac{2}{2}\right)\\ &= \frac{7(2)}{9(2)}\\ &= \frac{14}{18}.\end{aligned}$$

Then

$$\begin{aligned}\frac{5}{6} + \frac{7}{9} &= \frac{15}{18} + \frac{14}{18}\\ &= \frac{29}{18}.\end{aligned}$$

Example. Find a common name for $\dfrac{\frac{2}{3} + 5}{\frac{3}{7}}$.

$$\frac{\frac{2}{3} + 5}{\frac{3}{7}} = \frac{\frac{2}{3} + 5}{\frac{3}{7}} \cdot \frac{21}{21} \quad \text{(Why did we use } \tfrac{21}{21}\text{?)}$$

$$= \frac{(\frac{2}{3} + 5)21}{(\frac{3}{7})21}$$

$$= \frac{\frac{2}{3}(21) + 5(21)}{\frac{3}{7}(21)} \quad \text{(Note use of the distributive property.)}$$

$$= \frac{14 + 105}{9}$$

$$= \frac{119}{9}.$$

Problems

Show how you use the properties of 0 and 1 to find a common name for each of the following:

1. $\frac{1}{2} + \frac{1}{3}$

2. $\frac{7}{12} + \frac{5}{18}$

3. $\dfrac{7 + \frac{2}{3}}{\frac{5}{6}}$

4. $\dfrac{\frac{1}{2} + \frac{3}{5}}{\frac{3}{20}}$

5. (a) If you know that the product of two numbers is 0, and that one of the numbers is 3, what can you tell about the other number?

   (b) If the product of two numbers is 0, what can you tell about at least one of the numbers?

   (c) Does the multiplication property of 0 provide answers to these questions? Is another property of 0 implied here?

Closure. In our work so far we have often combined two numbers by addition or multiplication to obtain a number. We have never doubted that we always *do* get a number because our experience is that we always do.

If you add any two of the numbers of arithmetic the sum is always a number of this set. When a certain operation is performed on elements of a given subset of the numbers of arithmetic and the resulting number is always a member of the same subset, then we say that the *subset is closed under the operation*. We say, therefore, that the set of numbers of arithmetic is *closed under addition*. Likewise, since the product of any two numbers is always a number, the set of numbers is *closed under multiplication*. We state these properties in the language of algebra as follows:

Closure Property of Addition: For every number a and every number b, $a + b$ is a number.

Closure Property of Multiplication: For every number a and every number b, ab is a number.

Associative and Commutative Properties of Addition and Multiplication. The algebraic language with which we have been becoming familiar permits us, as in the case of the properties of 1 and 0 which we have just studied, to give a statement about the above property in this language. We have three (not necessarily different) numbers to deal with at once. Let us call the first "a", the second "b", and the third "c". "Add a second number to a first number" is then interpreted as "$a + b$"; "add a third number to their sum" is interpreted as "$(a + b) + c$". (Why did we insert the parentheses?) Write the second half of our verbal statement in the language of algebra. The words "the outcome is the same" now tell us that we have two names for the same number. Our statement becomes

$$(a + b) + c = a + (b + c).$$

For what numbers is this sentence true? We have concluded previously that it is true for all numbers. And so we write, finally,

For every number a, for every number b, for every number c,

$$(a + b) + c = a + (b + c).$$

We recall the commutative property of addition. It was verbalized as follows: If two numbers are added in different orders, the results are the same. In the language of algebra, we say

For every number a and every number b,

$$a + b = b + a.$$

How would you state the associative property of multiplication in the language of algebra?

What property is given by the following statement?

For every number a and every number b,

$$ab = ba.$$

These properties of the operations enable us to write open phrases in "other forms". For example, the open phrase $3d(d)$ can be written in the form $3(d \cdot d)$, i.e., $3d^2$, by applying the associative property of multiplication. Thus, two "forms" of an open phrase are two numerals for the same number.

Among the properties with which we have just been concerned are the commutativity of addition and multiplication. Why are we so concerned whether binary operations like addition and multiplication are commutative? Aren't all the operations of arithmetic commutative? Let us try division, for example. Recall that

$$6 \div 3$$

means "6 divided by 3". Now, test whether

$$6 \div 3 = 3 \div 6$$

is a true sentence. This is enough evidence to show that division is not a commutative operation. (By the way, can you find some a and some b such that $a \div b = b \div a$?) Is the division operation associative?

Another very interesting example for the counting numbers is the following: let 2**3 be defined to mean (2)(2)(2); and 3**2 to mean (3)(3). In general, a**b means a has been used as a factor b times. Is the following sentence true?

$$5{*}{*}2 = 2{*}{*}5?$$

Do you conclude that this binary operation on counting numbers is commutative? Is it associative?

You may complain that this second example is artificial. On the contrary, the ** operation defined above is actually used in the language of certain digital computers. You see, a machine is much happier if you give it all its instructions on a line, and so a "linear" notation was devised for this operation. But you see that to the machine the order of the numbers makes a great difference in this operation. Is there any restriction on the types of numbers on which we may operate with **?

Problems

1. If x and y are numbers of arithmetic, the closure property assures us that $3xy$, $2x$ and therefore, $(3xy)(2x)$ are numbers of arithmetic. Then the associative and commutative properties of multiplication enable us to write this in another form:

   $$\begin{aligned}(3xy)(2x) &= (3\cdot 2)(x\cdot x)y\\ &= 6x^2y.\end{aligned}$$

   Write the indicated products in another form as in the above example:

   (a) $(2m)(mn)$
   (b) $(5p^2)(3q)$
   (c) $n(2n)(3m)$
   (d) $(\frac{1}{2}ab)(6c)$
   (e) $(10a)(10b)$
   (f) $(3x)(12)$

2. If x and y are numbers of arithmetic then the closure property allows us to think of $12x^2y$ as a numeral which represents a single number. The commutative and associative properties of multiplication enable us to write other numerals for the same number. $(4xy)(3x)$, $(2x)(6xy)$, and $(12x^2y)(1)$ are some of the many ways of writing $12x^2y$ as indicated products. Similarly, write three possible indicated products for each of the following.

   (a) $8ab^2$
   (b) $7xy^2$
   (c) $10mn$
   (d) x^2y^2
   (e) $64a^2bc^2$
   (f) $2c$

3. Which of the following sentences are true for every value of the variable? Explain which of the properties aided in your decision.

   (a) $m(2 + 5) = (2 + 5)m$

   (b) $(m + 1)2 = (2 + 1)m$

   (c) $(a + 2y) + b = (a + b) + 2y$

   (d) $3x + y = y + 3x$

   (e) $(2uv)z = 2u(vz)$

4. The set A is given by $A = \{0, 1\}$.

   (a) Is A closed under addition?

   (b) Is A closed under multiplication?

5. (a) Is the set S of all multiples of 6 closed under addition?

   (b) Is set S closed under multiplication?

6. Let us define some binary operations other than addition and multiplication. We shall use the symbol "o" each time. We might read "a o b" as "a operation b". Since we are giving the symbol various meanings, we must define its meaning each time. For instance, for every a and every b,

   if $a \circ b$ means $2a + b$, then $3 \circ 5 = 2(3) + 5$;

   if $a \circ b$ means $\frac{a + b}{2}$, then $3 \circ 5 = \frac{3 + 5}{2}$;

   if $a \circ b$ means $(a - a)b$, then $3 \circ 5 = (3 - 3)5$;

   if $a \circ b$ means $a + \frac{1}{3}b$, then $3 \circ 5 = 3 + (\frac{1}{3})(5)$;

   if $a \circ b$ means $(a + 1)(b + 1)$, then
   $3 \circ 5 = (3 + 1)(5 + 1)$.

   For each meaning of $a \circ b$ stated above, write a numeral for each of the following:

   (a) $2 \circ 6$

   (b) $(\frac{1}{2}) \circ 6$

   (c) $6 \circ 2$

   (d) $(3 \circ 2) \circ 4$

7. Are such binary operations on numbers as those defined in Problem 6 commutative? In other words, is it true that for every a and every b, $a \circ b = b \circ a$? Let us examine some cases. For instance, if $a \circ b$ means $2a + b$, we see that

$$3 \circ 4 = 2(3) + 4$$
$$4 \circ 3 = 2(4) + 4.$$

But "$2(3) + 4 = 2(4) + 3$" is a false sentence. Hence, we conclude that the operation here indicated by "o" is not commutative. In each of the following, decide whether or not the operation described is commutative:

(a) For every a and every b, $a \circ b = \frac{a + b}{2}$

(b) For every a and every b, $a \circ b = (a - a)b$

(c) For every a and every b, $a \circ b = a + \frac{1}{3}b$

(d) For every a and every b, $a \circ b = (a + 1)(b + 1)$.

What do you conclude about whether all binary operations are commutative?

8. Is the operation "o" associative in each of the above cases? For instance, if, for every a and every b, $a \circ b = 2a + b$, is $(4 \circ 2) \circ 5 = 4 \circ (2 \circ 5)$ a true sentence?

$$\begin{aligned}(4 \circ 2) \circ 5 &= 2\big(2(4) + 2\big) + 5\\ &= 2(10) + 5\end{aligned}$$

while

$$\begin{aligned}4 \circ (2 \circ 5) &= 2(4) + \big(2(2) + 5\big)\\ &= 8 + 9.\end{aligned}$$

Since the sentence $2(10) + 5 = 8 + 9$ is false, we conclude that this operation is not associative. Test the operations described in Problem 7 (a)-(d) for the associative property.

The Distributive Property. Our previous work with numbers has shown us a variety of versions of the distributive property. Thus,

$$15(7 + 3) = 15(7) + 15(3)$$

and

$$(\frac{1}{3})12 + (\frac{1}{4})12 = (\frac{1}{3} + \frac{1}{4})12$$

are two true sentences each of which follows one of the patterns which we have recognized. We have seen the importance of this property in relating indicated sums and indicated products. We may now state the *distributive property* in the language of algebra:

For every number a, every number b, and every number c,

$$a(b + c) = ab + ac.$$

Since we have stated that "$a(b + c)$" and "$ab + ac$" are numerals for the same number, we may equally well write

For every number a, every number b, and every number c,

$$ab + ac = a(b + c).$$

We may also apply the commutative property of multiplication to write:

For every number a, every number b, and every number c,

$$(b + c)a = ba = bc$$

and:

For every number a, every number b, and every number c,

$$ba + ca = (b + c)a.$$

Any one of the four sentences above describes the distributive property. All forms are useful in the study of algebra.

Example 1. Write the indicated product, $x(y + 3)$ as an indicated sum.

$$x(y + 3) = xy + x(3) \quad \text{by the distributive property}$$

$$= xy + 3x$$

Example 2. Write $5x + 5y$ as an indicated product.

$5x + 5y = 5(x + y)$ by the distributive property

Example 3. Write the open phrase $3a + 5a$ in simpler form.

$3a + 5a = (3 + 5)a$ by the distributive property

$= 8a$

Example 4. Write the open phrase, $2x + 3y + 4x + 6y$ in simpler form.

$2x + 3y + 4x + 6y = (2x + 4x) + (3y + 6y)$

by the commutative and associative properties of addition.

$= (2 + 4)x + (3 + 6)y$

by the distributive property

$= 6x + 9y$

Problems

1. Write the indicated products as indicated sums.

(a) $6(r + s)$

(b) $(b + 3)a$

(c) $x(x + z)$

(d) $(7 + x)x$

(e) $6(8 + 5)$

(f) $(a + b)b$

2. Write the indicated sums as indicated products.

(a) $3x + 3y$

(b) $am + an$

(c) $x + bx$ Hint: $x = (1)x$

(d) $\frac{1}{2}x + \frac{1}{2}y$

(e) $2a + a^2$ Hint: How is a^2 defined?

(f) $x^2 + xy$

3. Use the associative, commutative, and distributive properties to write the following open phrases in simpler form, if possible:

(a) $14x + 3x$

(b) $\frac{3}{4}x + \frac{3}{2}x$

(c) $\frac{2}{3}a + 3b + \frac{1}{3}a$

(d) $7x + 13y + 2x + 3y$

(e) $4x + 2y + 2 + 3x$

(f) $1.3x + 3.7y + 6.2 + 7.7x$

(g) $2a + \frac{1}{3}b + 5$

The distributive property stated by the sentence,

For every number a, every number b, and every number c,

$$a(b + c) = ab + ac$$

concerns the three numbers a, b and c. However, the closure property allows us to apply the distributive property in many cases where an open phrase apparently contains more than three numerals. For example, suppose we wish to express the indicated product $2r(s + t)$ as a sum. The open phrase contains the four numerals 2, r, s, and t. The closure property, however, allows us to consider $2r$ as the name of one number so we can think in terms of three numerals, $2r$, s, and t. Thus,

$$\begin{aligned} 2r(s + t) &= (2r)(s + t) \\ &= (2r)s + (2r)t \\ &= 2rs + 2rt \end{aligned}$$

Example 1. Write $3u(v + 3z)$ as an indicated sum.

By the closure property we can regard $3u$, v, and $3z$ each as the name of one number. Then by the distributive property

$$\begin{aligned} 3u(v + 3z) &= (3u)v + (3u)(3z) \\ &= 3uv + 9uz \end{aligned}$$

by the commutative and associative properties of multiplication.

Example 2. Write the indicated sum, $2rs + 2rt$, as an indicated product.

We can do this in three ways:

(1) $$\begin{aligned} 2rs + 2rt &= 2(rs) + 2(rt) \\ &= 2(rs + rt) \end{aligned}$$

(2) $$\begin{aligned} 2rs + 2rt &= r(2s) + r(2t) \\ &= r(2s + 2t) \end{aligned}$$

(3) $$\begin{aligned} 2rs + 2rt &= (2r)s + (2r)t \\ &= 2r(s + t) \end{aligned}$$

Although all three ways are correct, the third is usually preferred.

Example 3. Express the indicated product, $3(x + y + z)$, as an indicated sum.

$$3(x + y + z) = 3x + 3y + 3z.$$

Problems

1. Write each of the indicated products as an indicated sum.

(a) $m(6 + 3p)$

(b) $2k(k + 1)$

(c) $6(2s + 3r + 7q)$

(d) $(2x + xy)x$

(e) $(e + f + g)h$

(f) $6pq(p + q)$

2. Which of the following open sentences are true for every value of every variable. (Hint: Use the commutative, associative and distributive properties to write both members of these sentences in the same form.)

(a) $2a(a + b) = 2a^2 + ab$

(b) $4xy + y^2 = (4x + y)y$

(c) $3ab + 6bc = 3b(a + 2c)$

(d) $2a(b + c) = 2ab + c$

(e) $(4x + 3)x = 4x^2 + 3 + x$

(f) $(2y + xy) = (2 + x)y$

3. Write each of the indicated sums as an indicated product.

   (a) $3uv + v^2$

   (b) $7pq + 7qr$

   (c) $3x + 3x^2$ Hint: Think of $3x$ as $(3x)(1)$

   (d) $2c + 4cd$ Hint: $4cd = (2c)(2d)$

   (e) $3x + 6x^2$

   (f) $xz^2 + 2xz$

Another important application of the distributive property is illustrated by the following example.

Example 1. Write $(x + 2)(x + 3)$ as an indicated sum without parentheses.

If we write the distributive property with the indicated product beneath it, we can see which names we must regard as separate names of numbers.

$$\begin{array}{ccccc} a(b + c) & = & ab & + & ac \\ \downarrow \quad \searrow \quad \searrow & & \swarrow \; \downarrow & & \swarrow \; \downarrow \\ \overbrace{(x + 2)}(x + 3) & = & \overbrace{(x + 2)}x & + & \overbrace{(x + 2)}3 \end{array}$$

$$\begin{aligned} &= x^2 + 2x + 3x + 6 && \text{distributive property} \\ &= x^2 + (2 + 3)x + 6 && \text{distributive property} \\ &= x^2 + 5x + 6. \end{aligned}$$

Could you have used a different form of the distributive property to begin your work?

Example 2. Write $(a + b)(c + d)$ as an indicated sum without parentheses. Supply the reason for each step.

$$\begin{aligned} (a + b)(c + d) &= (a + b)c + (a + b)d \\ &= ac + bc + ad + bd \end{aligned}$$

Problems

Write the indicated products as indicated sums without parentheses.

1. $(x + 4)(x + 2)$
2. $(x + 1)(x + 5)$
3. $(x + a)(x + 3)$
4. $(x + 2)(y + 7)$
5. $(m + n)(m - n)$
6. $(2p + q)(p + 2q)$
7. List all the properties of operations on numbers of arithmetic.
8. Look for the pattern in the following calculation:

$$\begin{aligned} 19 \times 13 &= 19(10 + 3) \\ &= 19(10) + 19(3) && \text{(what property?)} \\ &= 19(10) + (10 + 9)3 \\ &= 19(10) + \big(10(3) + 9(3)\big) && \text{(what property?)} \\ &= \big(19(10) + 10(3)\big) + 9(3) && \text{(what property?)} \\ &= (19 + 3)10 + 9(3) && \text{(what properties?)} \end{aligned}$$

The final result indicates a method for "multiplying teens" (whole numbers from 11 through 19): Add to the first number the units digit of the second, and multiply by 10; then add to this the product of the units digits of the two numbers. Use the method to find 15×14, 13×17, 11×12.

9. Find the coordinate of a point which lies on the number line between the two points with coordinates $\frac{3}{4}$ and $\frac{5}{6}$. How many points are between these two?
10. Consider the set

$$T = \{0, 3, 6, 9, 12, \ldots\}.$$

Is T closed under the operation of addition? Under the operation of "averaging"?

11. Consider the open sentence

$$2x \leq 1.$$

What is its truth set if the domain of x is the set of:

(a) all counting numbers?

(b) all whole numbers?

(c) all numbers of arithmetic?

12. Explain how the property of 1 is used in performing the calculation

$$\frac{\frac{3}{5} + \frac{2}{3}}{\frac{3}{4}}$$

13. Explain why

$$3x + y + 2x + 3y = 5x + 4y$$

is true for all values of x and y.

14. (a) Write the indicated products

$$(x + 1)(x + 1)$$
$$(x + 2)(x + 2)$$

as indicated sums without parentheses.

(b) Use the pattern of the results of Part (a) to write the indicated sum

$$x^2 + 6x + 9$$

as an indicated product.

Part 2

OPEN SENTENCES AND ENGLISH SENTENCES

Open Phrases and English Phrases. Every day we are building up a new language of symbols which is becoming more and more a complete language. We have used mathematical phrases, such as "$8 + 3y$"; mathematical verb forms, including "$=$" and "$>$"; and mathematical sentences, such as "$7n + 3n = 50$".

We recall that a variable, such as "n", is the name of a definite but unspecified number. The translation of "n" into English will then mean relating an unspecified number to something of interest to us. Thus, the numeral "n" might represent "the number of problems that I worked", "the number of students at the rally", "the number of dimes in Sam's pocket", or "the number of feet in the height of the school flagpole". What are some other possible translations?

Consider the phrase "$5 + n$". Can we invent an English phrase for this? Suppose we use the translations suggested above. If "n" is the number of problems I shall be working today, then the phrase "$5 + n$" represents "the total number of problems including the five worked last night"; or, if I have 5 dimes and "n" represents the number of dimes in Sam's pocket, then "$5 + n$" represents "the total number of dimes, including my five and those in Sam's pocket." Notice that the translation of "$5 + n$" depends on what translation we make of "n".

Which of the apparently limitless number of translations do we pick? We are reminded that the variable appearing in the open phrase, whether "n" or "x", or "w", or "b", is the name of a number. Whether this is the number of dimes, the number of students, the number of inches, etc., depends upon the use we plan to make of the translation. The context itself will frequently suggest or limit translations. Thus it would not make sense to translate a phrase such as "$2{,}500{,}000 + y$" in terms of the number of dimes in Sam's pocket, but it would make sense to think of "y" as representing the number giving the population increase in a state which had 2,500,000 persons at the time of the preceding census, or as the number of additional miles traveled by a satellite which had gone 2,500,000 miles at the time of the last report. Similarly, the variable in the phrase "$.05 + k$" would hardly be translated as the number of cows or students, but possibly as the number giving the increase in the rate of interest which had previously been 5 per cent.

How can we translate the phrase "$3x + 25$"? In the absence of any special reasons for picking a particular translation, we might let x be the number of cents Tom earns in one hour, mowing the lawn. Then $3x$ is the number of cents earned in 3 hours. If Tom finished the job in three hours and was paid a bonus of 25 cents, then the phrase "$3x + 25$" represents the total number of cents in Tom's possession after working three hours. How can this phrase be translated if we let x be the number of students in each algebra class, if algebra classes are of the same size? Or, if x is the number of miles traveled by a car in one hour at a constant speed?

There are many English translations of the symbol "+", indicating the operation of addition of two numbers. A few of them are: "the sum of", "more than", "increased by", "older than", and others. There are also many English translations of the symbols indicating the operation of multiplication of two numbers, including: "times", "product of", and others. What are English translations of the symbol "-"?

Problems

In Problems 1-6, write English phrases which correspond to the given open phrases. Try to vary the English phrases as much as possible. Tell in each case what the variable represents.

1. $7w$ (If one bushel of wheat costs w dollars, the phrase is: "the number of dollars in the cost of 7 bushels of wheat".)

2. $n + 7$

3. $n - 7$

4. $\frac{y}{2}$

5. $2r + 5$

6. $a + b$

In each of Problems 7-14, find an open phrase which is a translation of the given English phrase. In each problem, tell explicitly what the variable represents.

7. The number of feet in y yards.
(If y is the number of yards, then $3y$ is the number of feet.)

8. The number of inches in f feet.

9. The number of pints in k quarts.

10. The successor of a whole number.

11. The reciprocal of a number. (Two numbers are reciprocals of each other if their product is 1.)

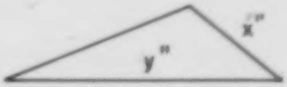
12. The number of ounces in k pounds and t ounces.

13. The number of cents in m dollars, k quarters, m dimes and n nickels.

14. The number of inches in the length of a rectangle which is twice as long as it is wide. (Suggestion: Draw a figure to help visualize the situation.)

15. Choose a variable for the number of feet in the length of one side of a square. Write an open phrase for the number of feet in the perimeter of the square.

16. One side of a triangle is x inches long and a second is y inches long. The length of the third side is one-half the sum of the lengths of the first two sides.



(a) Write an open phrase for the number of inches in the perimeter of the triangle.

(b) Write an open phrase for the number of inches in the length of the third side.

17. The admission price to a performance of "The Mikado" is $2.00 per person. Write an open phrase for the total number of dollars received in terms of the number of people who bought tickets.

18. If a man can paint a house in d days, write an open phrase for the part of the house he can paint in one day.

19. If a pipe fills $\frac{1}{5}$ of a swimming pool in one hour, write an open phrase for how much of the pool is filled by that pipe in x hours.

20. When a tree grows it increases its radius each year by adding a ring of new wood. If a tree has r rings now, write an open phrase for the number of growth rings in a tree twelve years from now.

21. A plant grows a certain number of inches per week. It is now 20 inches tall. Write an open phrase giving the number of inches in its height five weeks from now.

22. Choose a variable for the number of feet in the width of a rectangle.

    (a) Write an open phrase for the length of the rectangle if the length is five feet less than twice the width. Draw and label a figure.

    (b) Write an open phrase for the perimeter of the rectangle described in Part (a).

    (c) Write an open phrase for the area of the rectangle described in Part (a).

Open Sentences and English Sentences. Often we want to translate English sentences into open sentences. We find such open sentences particularly helpful in word problems when the English sentence is about a quantity which we are interested in finding.

Example 1. "Carl has a board 44 inches long. He wishes to cut it into two pieces so that one piece will be three inches longer than the other. How long should the shorter piece be?"

We may sometimes see more easily what our open sentence should be if we guess a number for the quantity asked for in the problem.

If the shorter piece is 18 inches long, then the longer piece is $(18 + 3)$ inches long. Since the whole board is 44 inches long, we then have the sentence

$$18 + (18 + 3) = 44.$$

Although this sentence is not true, it suggests the pattern which we need for an open sentence. Notice that the question in the problem has pointed out our variable. We can now say:

> If the shorter piece is K inches long, then the longer piece is $(K + 3)$ inches long, and the sentence is

$$K + (K + 3) = 44.$$

We say that this sentence is false when K is 18. There probably is some value of K for which the open sentence is true. If we wanted to find the length of the shorter piece, this could be done by finding the truth set of the above open sentence.

Notice that the English sentences are often about inches or pounds or years or dollars, but the open sentences are always about numbers only.

Notice also that we are very careful in describing our variable to show what it measures, whether it is the number of inches, the number of donkeys, or the number of tons.

Example 2. "Two cars start from the same point at the same time and travel in the same direction at constant speeds of 34 and 45 miles per hour, respectively. In how many hours will they be 35 miles apart?"

If they travel 4 hours, the faster car goes $45(4)$ miles and the slower car goes $34(4)$ miles. Since the faster car should then be 35 miles farther from the starting point than the slower car, we have the sentence

$$45(4) - 34(4) = 35,$$

which is false. It suggests, however, the following:

> If they travel h hours, then the faster car goes $45h$ miles and the slower car goes $34h$ miles, and
>
> $$45h - 34h = 35.$$

Example 3. "A man left $10,500 for his widow, a son and a daughter. The widow received $5,000 and the daughter received twice as much as the son. How much did the son receive?"

> If the son received n dollars, then the daughter received $2n$ dollars, and
>
> $$n + 2n + 5000 = 10,500.$$

Problems

Write open sentences that would help you solve the following problems, being careful to give the meaning of the variable for each. Your work may be shown in the form indicated in Example 3 above. Do not find the truth sets of the open sentences.

1. A rectangle is 6 times as long as it is wide. Its perimeter is 144 inches. How wide is the rectangle? (Remember to draw a figure.)

2. The largest angle of a triangle is 20° more than twice the smallest, and the third angle is 70°. The sum of the angles of a triangle is 180°. How large is the smallest angle?

3. A class of 43 students was separated into two classes. If there were 5 more students in Mr. Smith's class than in Miss. Jones's class, how many students were in each class? (Can you do this one in two ways? If there were y students in Miss. Jones's class, find two ways to say how many were in Mr. Smith's class.)

4. John is three times as old as Dick. Three years ago the sum of their ages was 22 years. How old is each now? (Hint: Find a phrase for the age of each three years ago in terms of Dick's age now.)

5. John has $1.65 in his pocket, all in nickels, dimes, and quarters. He has one more quarter than he has dimes, and the number of nickels he has is one more than twice the number of dimes. How many dimes has he? (Hint: If he has d dimes, write a phrase for the value of all his dimes, a phrase for the value of all his quarters, and a phrase for the value of all his nickels; then write your open sentence.)

6. A passenger train travels 20 miles per hour faster than a freight train. At the end of 5 hours the passenger train has traveled 100 miles farther than the freight train. How fast does the freight train travel? (Hint: For each train find a phrase for the number of miles it has traveled.)

7. Mr. Brown is employed at an initial salary of $3600, with an annual increase of $300, while Mr. White starts at the same time at an initial salary of $4500, with an annual increase of $200. After how many years will the two men be earning the same salary?

8. A table is three times as long as it is wide. If it were 3 feet shorter and 3 feet wider, it would be a square. How long and how wide is it? (Draw two pictures of the table top.)

Open Sentences Involving Inequalities. Our sentences need not all be equalities. Problems concerning "greater than" or "less than" have real meaning.

Suppose we say, "Make a problem for the sentence $d + 2 > 5$". The word problem could be, "If I added two dollars to what I now have, I would have more than five dollars. How much do I have now"?

As with equations, it will sometimes help to find an open sentence in problems about inequalities if we try a particular number first.

Example 1. "In six months Mr. Adams earned more than \$7000. How much did he earn per month?"

If he earned \$1100 per month, in 6 months he would earn 6×1100 dollars. The sentence would then be

$$6 \times 1100 > 7000.$$

This, of course, is not true, but it suggests what we should do.

If Mr. Adams earned a dollars per month, in 6 months he would earn $6a$ dollars. Then

$$6a > 7000.$$

Example 2. "The distance an object falls during the first second is 32 feet less than the distance it falls during the second second. During the two seconds it falls 48 feet or less, depending on the air resistance. How far does it fall during the second second?

If the object falls 42 feet during the second second, then it falls $(42 - 32)$ feet during the first second. Since the total distance fallen is less than or equal to 48 feet, our sentence is

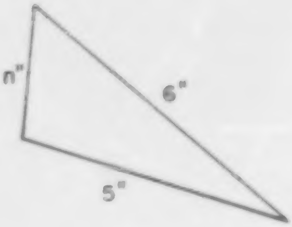
$$(42 - 32) + 42 \leq 48.$$

This suggests how to write the open sentence. If the object falls d feet during the second second, then it falls $(d - 32)$ feet during the first second, and

$$(d - 32) + d \leq 48.$$

Example 3. "Two sides of a triangle have lengths of 5 inches and 6 inches. What is the length of the third side?"

You may have drawn many triangles in the past and have become aware of the fact that the length of any side of a triangle must be less than the sum of the lengths of the other two sides. Thus, if the third side of this triangle is n inches long,



$$n < 5 + 6.$$

At the same time the six inch side must be less in length than the sum of the lengths of the other two; thus,

$$6 < n + 5.$$

Since both of these conditions must hold, the open sentence for our problem is

$$n < 5 + 6 \text{ and } 6 < n + 5.$$

Problems

Write open sentences for the following problems, being careful to give the meaning of the variable for each. Do not find the truth sets of the open sentences.

1. One third of a number added to three-fourths of the same number is equal to or greater than 26. What is the number?
2. Bill is 5 years older than Norman, and the sum of their ages is less than 23. How old is Norman?

3. A square and an equilateral triangle have equal perimeters. A side of the triangle is five inches longer than a side of the square. What is the length of the side of the square? Draw a figure.

4. A boat, traveling downstream, goes 12 miles per hour faster than the rate of the current. Its velocity downstream is less than 30 miles per hour. What is the rate of the current?

5. On a half-hour TV show the advertiser insists there must be at least three minutes for commercials and the network insists there must be less than 12 minutes for commercials. Express this in a mathematical sentence. How much time must the program director provide for material other than advertising?

6. A student has test grades of 75 and 82. What must he score on a third test to have an average of 88 or higher? If 100 is the highest score possible on the third test, how high an average can he achieve? What is the lowest average he can achieve?

7. Using two variables, write an open sentence for each of the following English sentences.

   (a) The enrollment in Scott School is greater than the enrollment in Morris School.

   (b) The enrollment in Scott School is 500 greater than the enrollment in Morris School.

Review Problems

1. Write open sentences corresponding to the following word sentences, using one variable in each.

   (a) The sum of a whole number and its successor is 575.

   (b) The sum of a whole number and its successor is 576.

   (c) The sum of two numbers, the second greater than the first by 1, is 576.

(d) A board 16 feet long is cut in two pieces such that one piece is one foot longer than twice the other.

(e) Catherine earns $2.25 baby-sitting for 3 hours at x cents an hour.

2. A two-digit number is 7 more than 3 times the sum of the digits. Restate this by an open sentence. (Hint: Express the number by means of two variables.)

3. The sum of two numbers is 42. If the first number is represented by n, write an expression for the second number using the variable n.

4. (a) A number is increased by 17 and the sum is multiplied by 3. Write an open sentence stating that the resulting product equals 192.

   (b) If 17 is added to a number and the sum is multiplied by 3, the resulting product is less than 192. Restate this as an open sentence.

5. One number is 5 times another. The sum of the two numbers is 15 more than 4 times the smaller. Express this by an open sentence.

6. (a) A farmer can plow a field in 7 hours with one of his tractors. How much of the field can he plow in one hour with that tractor?

   (b) With his other tractor he can plow the field in 5 hours. If he had both tractors going for 2 hours, how much of the field would be plowed?

   (c) How much of the field would then be left unplowed?

   (d) Write an open sentence which indicates that, if both tractors are used for x hours, the field will be completely plowed.

7. Mr. Brown is reducing. During each month for the past 8 months he has lost 5 pounds. His weight is now 175 pounds. What was his weight m months ago if $m < 8$? Write an open sentence stating that m months ago his weight was 200 pounds.

Write open sentences for Problems 8 to 13. Tell clearly what the variable represents, but do not find the truth set of the open sentence.

8. (a) The sum of a whole number and its successor is 45. What are the numbers?

   (b) The sum of two consecutive odd numbers is 76. What are the numbers?

9. Mr. Barton paid \$176 for a freezer which was sold at a discount of 12% of the marked price. What was the marked price?

10. A man's pay check for a week of 48 hours was \$166.40. He is paid at the rate of $1\frac{1}{2}$ times his normal rate for all hours worked in excess of 40 hours. What is his hourly pay rate?

11. (a) At an auto parking lot, the charge is 35 cents for the first hour, or fraction of an hour, and 20 cents for each succeeding (whole or partial) one-hour period. If t is the number of one-hour periods parked after the initial hour, write an open phrase for the parking fee.

    (b) With the same charge for parking as in the preceding problem, if h is the total number of one-hour periods parked, write an open phrase for the parking fee.

12. Two quarts of alcohol are added to the water in the radiator, and the mixture then contains 20 per cent alcohol; that is, 20 per cent of the mixture is pure alcohol. Write an open sentence for this English sentence. (Hint: Write an open phrase for the number of quarts of alcohol in terms of the number of quarts of water originally in the radiator.)

13. (a) Two water-pipes are bringing water into a reservoir. One pipe has a capacity of 100 gallons per minute, and the second 40 gallons per minute. If water flows from the first pipe for x minutes and from the second for y minutes, write an open phrase for the total flow in gallons.

(b) In the preceding problem, if the flow from the first pipe is stopped at the end of two hours, write the expression for the total flow in gallons in y minutes, where y is greater than 120.

(c) With the data in Part (a), write an open sentence stating that the total flow is 20,000 gallons.

14. A man, with five dollars in his pocket, stops at a candy store on his way home with the intention of taking his wife two pounds of candy. He finds candy by the pound box selling for $1.69, $1.95, $2.65, and $3.15. If he leaves the store with two one-pound boxes of candy;

(a) What is the smallest amount of change he could have?

(b) What is the greatest amount of change he could have?

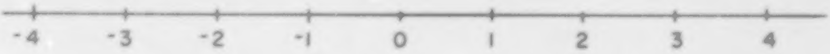
(c) What sets of two boxes can he not afford?

Part 3

THE REAL NUMBERS

The Real Number Line. We know from past experience that there are rational numbers to be associated with points on the left half of the number line, but meanwhile we have dealt only with rational numbers on the right half. For another thing, we know that some points on the number line do not correspond to rational numbers. Where are some of these points on the number line which do not correspond to rational numbers, and what new numbers are associated with them?*

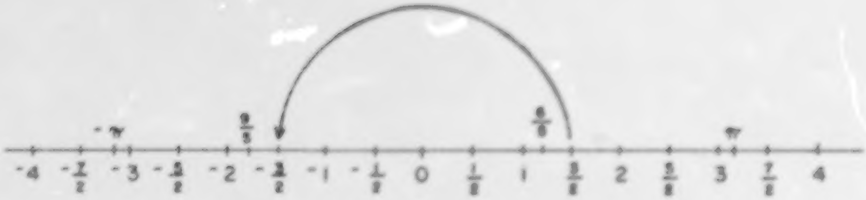
How shall we label the points on the left of 0? There is no doubt that the line contains infinitely many points to the left of 0. It is an easy matter to label such points if we follow the pattern we used to the right of 0. As before, we use the interval from 0 to 1 as the unit of measure, and locate points equally spaced along the line *to the left*. The first of these we label $^{-}1$, the second $^{-}2$, etc., where the symbol "$^{-}1$" is read "*negative* 1", $^{-}2$ is read "negative 2", etc. What is the coordinate of the point which is 7 units to the left of 0?



Proceeding as before, we can find additional points to the left of 0 and label them with symbols similar to those used for numbers to the right, with an upper dash to indicate that the number is to the left of 0. Thus, for example, $\frac{^{-}3}{2}$ is the same distance from 0 on the left as $\frac{3}{2}$ is on the right, etc.

---

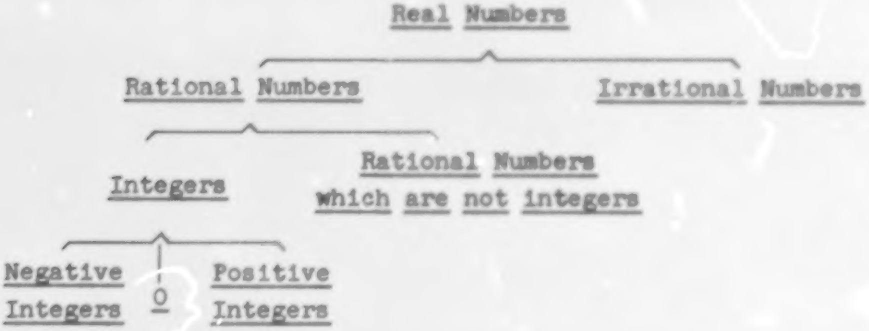
*It is assumed here that the reader is familiar with Studies in Mathematics, Vol. 1, "Number Systems".


-π
9/5
6/5
π
-4
-7/2
-3
-5/2
-2
-3/2
-1
-1/2
0
1/2
1
3/2
2
5/2
3
7/2
4

The set of *all* numbers associated with points on the number line is called the set of *real numbers*. The numbers to the left of zero are called the *negative real numbers* and those to the right are called the *positive real numbers*. In this language, the numbers of arithmetic are the *non-negative real numbers*.

The set of all whole numbers $\{0, 1, 2, 3, \ldots\}$ combined with the set $\{^{-}1, {}^{-}2, {}^{-}3, \ldots\}$ is called the set of *integers* $\{\ldots, {}^{-}3, {}^{-}2, {}^{-}1, 0, 1, 2, 3, \ldots\}$. The set of all rational numbers of arithmetic combined with the negative rational numbers is called the set of *rational numbers*. (Certainly, all rational numbers are real numbers.)

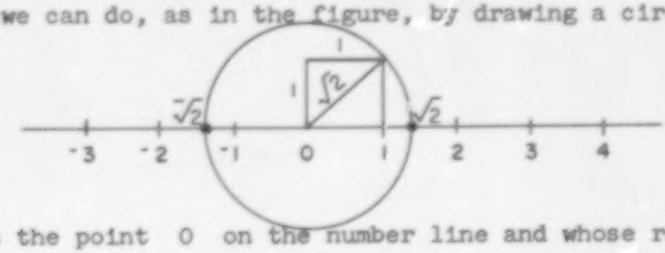
Remember that each rational number is now assigned to a point of the number line, but *there remain many points to which rational numbers cannot be assigned*; in fact, there remain *more* points which have *not* yet been assigned numbers than there are presently associated with rational numbers. The numbers associated with these points are called the *irrational numbers*. (Thus, all irrational numbers are also real numbers.) Hence, we can regard the set of real numbers as the combined set of rational and irrational numbers.


Real Numbers
Rational Numbers
Irrational Numbers
Integers
Rational Numbers which are not integers
Negative Integers
0
Positive Integers

For example, all integers, such as $^{-}4$, 0, 2, are rational numbers; find examples of rational numbers which are not integers. Furthermore, all rational numbers such as $^{-}\frac{3}{2}$, 0, 6, are real numbers.

Where are some of the points on the number line which do not correspond to rational numbers? It will be proved in a later part that, for example, the real number $\sqrt{2}$ is an irrational number. Let us locate the points with coordinates $\sqrt{2}$ and $^{-}\sqrt{2}$, respectively.

First of all, we recall that $\sqrt{2}$ is a number whose square is 2. You may have learned that the length of a diagonal of a square, whose sides have length 1, is a number whose square is 2. (Do you know any facts about right triangles which will help you verify this?) In order to locate a point on the number line for $\sqrt{2}$, all we have to do is construct a square with side of length 1 and transfer the length of one of its diagonals to our number line. This we can do, as in the figure, by drawing a circle whose center is at the point 0 on the number line and whose radius is the same length as the diagonal of the square. This circle cuts the number line in two points, whose coordinates are the real numbers $\sqrt{2}$ and $^{-}\sqrt{2}$, respectively.



Later you will prove that the number $\sqrt{2}$ is not a rational number. Maybe you believe that $\sqrt{2}$ is 1.4. Test for yourself whether this is true by squaring 1.4. Is $(1.4)^2$ the same number as 2? In the same way, test whether $\sqrt{2}$ is 1.41; 1.414. The square of each of these decimals is closer to 2 than the preceding, but there seems to be no rational number whose square is 2.

There are many more points on the real number line which have coordinates which are not rational numbers. Do you think $\frac{1}{2}\sqrt{2}$ is such a point? $3 + \sqrt{2}$? Why?

Problems

1. Draw the graphs of the following sets:

   (a) $\{{}^{-}\frac{2}{3}, \frac{2}{3}, \frac{5}{2}, {}^{-}\frac{5}{2}\}$

   (b) $\{{}^{-}\frac{3}{2}, 5, {}^{-}7, {}^{-}\frac{11}{3}\}$

   (c) $\{\sqrt{2}, {}^{-}\sqrt{2}, 3, {}^{-}3\}$

   (d) $\{{}^{-}1, {}^{-}(1 + \frac{1}{2}), 1 + \frac{1}{2}\}$

   (e) $\{{}^{-}\frac{1}{2}, (\frac{1}{2})^{2}, {}^{-}\frac{6}{4}, (3 - 3)\}$

2. Of the two points whose coordinates are given, which is to the right of the other?

   (a) ${}^{-}\frac{5}{2}, 0$

   (b) ${}^{-}\frac{5}{2}, {}^{-}\frac{10}{4}$

   (c) $0, 3$

   (d) ${}^{-}4, \sqrt{2}$

   (e) ${}^{-}\frac{16}{3}, {}^{-}\frac{21}{4}$

   (f) ${}^{-}\frac{1}{2}, \frac{1}{2}$

3. The number π is the ratio of the circumference of a circle to its diameter. Thus, a circle whose diameter is of length 1 has a circumference of length π. Imagine such a circle resting on the number line at the point 0. If the circle is rolled on the line, without slipping, one complete revolution to the right, it will stop on a point. What is the coordinate of this point? If rolled to the left one revolution it will stop on what point? Can you locate these points approximately on the real number line? (The real number π, like $\sqrt{2}$, is not a rational number.)

4. (a) Is $^{-}2$ a whole number? An integer? A rational number? A real number?

   (b) Is $\frac{^{-}10}{3}$ a whole number? An integer? A rational number? A real number?

   (c) Is $^{-}\sqrt{2}$ a whole number? An integer? A rational number? A real number?

5. Which of the following sets are the same?
   A is the set of whole numbers, B is the set of positive integers, C is the set of non-negative integers, I is the set of integers, N is the set of counting numbers.

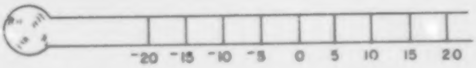
Order on the Real Number Line. How did we describe order for the positive real numbers? Since, for example, "5 is to the left of 6" on the number line, and since "5 is less than 6", we agreed that these two sentences say the same thing about 5 and 6. We wrote this as the true sentence

$$5 < 6.$$

Thus, for a pair of positive real numbers, "is to the left of" on the number line and "is less than" describe the same order.

What shall we mean by "is less than" for any two real numbers, whether they are positive, negative, or 0? Our answer is simply: "is to the left of" on the real number line.

Let us look for a justification in common experience. All of us are familiar with thermometers and are aware that scales on thermometers use numbers above 0 and numbers below 0, as well as 0 itself. We know that the cooler the weather, the lower on the scale we read the temperature. If we place a thermometer in a horizontal position, we see that it resembles part of our real number line. When we say "is less than" ("is a lower temperature than"), we mean "is to the left of" on the thermometer scale.



On this scale, which number is the lesser, $^{-}5$ or $^{-}10$?

Thus, we extend our former meaning of "is less than" to the whole set of real numbers. We agree that:

> "is less than" for real numbers means "is to the left of" on the real number line. If a and b are real numbers, "a is less than b" is written
>
> $$a < b.$$

(Now and in the future a variable is understood to have as its domain the set of real numbers, unless otherwise stated.)

Can you give a meaning for "is greater than" for real numbers? As before, use the symbol "$>$" for "is greater than". In the same way, explain the meanings of "$\leq$", "$\geq$", "$\nless$", "$\ngtr$" for real numbers.

Problems

1. For each of the following sentences, determine which are true and which false.

(a) $3 \leq {}^{-}1$

(b) $2 < \frac{^{-}7}{2}$

(c) $^{-}4 \nless 3.5$

(d) $\frac{^{-}12}{5} < {}^{-}2.2$

(e) $\frac{^{-}3}{5} \leq {}^{-}\left(\frac{3+0}{5}\right)$

(f) $^{-}4 \nleq 3.5$

(g) $^{-}6 > {}^{-}3$

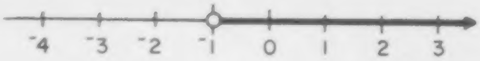
(h) $3.5 < {}^{-}4$

(i) $^{-}3 < {}^{-}2.8$

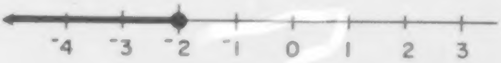
(j) $^{-}\pi \nless {}^{-}2.8$

2. Draw the graph of the truth set of each of the following sentences. For example:

$x > {}^{-}1$,



$x \leq {}^{-}2$,



(a) $y < 2$
(b) $u \nless 3$
(c) $v \geq \frac{{}^{-}3}{2}$
(d) $r \neq {}^{-}2$
(e) $x = 3$ or $x < {}^{-}1$
(f) $c < 2$ and $c > {}^{-}2$
(g) $a \leq {}^{-}3$ and $a \geq {}^{-}3$
(h) $d \leq {}^{-}1$ or $d > 2$
(i) $u > 2$ and $u < {}^{-}3$
(j) $a < 6$ and $a < {}^{-}2$

3. For each of the following sets, write an open sentence involving the variable x which has the given set as its truth set:

(a) A is the set of all real numbers not equal to 3.

(b) B is the set of all real numbers less than or equal to ${}^{-}2$.

(c) C is the set of all real numbers not less than $\frac{{}^{-}5}{2}$.

4. Choose any positive real number p; choose any negative real number n. Which, if any, of the following sentences are true?

$$n < p, \quad p < n, \quad n \leq p, \quad n \neq p.$$

5. Let the domain of the variable p be the set of integers. Then find the truth set of

(a) ${}^{-}2 < p$ and $p < 3$.

(b) $p \leq {}^{-}2$ and ${}^{-}4 < p$.

(c) $p = 2$ or $p = {}^{-}5$.

6. In the blanks below use one of $=$, $<$, $>$, to make a true sentence, if possible, in each case.

(a) $\frac{3}{5}$ ________ $\frac{^-6}{10}$

(b) $\frac{3}{5}$ ________ $\frac{3}{6}$

(c) $\frac{9}{12}$ ________ $\frac{8}{12}$

(d) $\frac{^-173}{29}$ ________ $\frac{^-183}{29}$

(e) $\frac{^-3}{5}$ ________ $\frac{3}{6}$

(f) $\frac{^-3}{5}$ ________ $\frac{^-3}{6}$

There are certain simple but highly important facts about the order of the real numbers on the real number line. If we choose any two different real numbers, we are sure that the first is less than the second or the second is less than the first, but not both. Stated in the language of algebra, this property of order for real numbers becomes the comparison property:

> If a is a real number and b is a real number, then exactly one of the following is true:
>
> $a < b$, $a = b$, $b < a$.

Problems

1. The comparison property stated in the text is a statement involving "$<$". Try to formulate the corresponding property involving "$>$".

2. Try to state a comparison property involving "$\geq$".

Which is less than the other, $\frac{4}{5}$ or $\frac{5}{6}$? You can find out by applying the multiplication property of 1 to each number to get $\frac{4}{5} = \frac{4}{5} \times \frac{6}{6} = \frac{24}{30}$ and $\frac{5}{6} = \frac{5}{6} \times \frac{5}{5} = \frac{25}{30}$. Then $\frac{4}{5} < \frac{5}{6}$, because $\frac{24}{30}$ is to the left of $\frac{25}{30}$ on the number line.

You should now be able to compare any two rational numbers. How would you decide which is the lesser, $\frac{337}{113}$ or $\frac{167}{55}$? (Describe the process; do not actually carry it out.)

Perhaps you noticed, in comparing $\frac{337}{113}$ and $\frac{167}{55}$, that $\frac{337}{113} < 3$ (i.e., $\frac{337}{113} < \frac{339}{113}$) and $3 < \frac{167}{55}$ (i.e., $\frac{165}{55} < \frac{167}{55}$). Could you now decide about the order of $\frac{337}{113}$ and $\frac{167}{55}$ without writing them as fractions with the same denominator? How could you find out similarly which is lesser, $\frac{40}{27}$ or $\frac{\pi}{2}$? Or suppose that x and y are real numbers and that $x < {}^-1$ and ${}^-1 < y$. Again using the number line, what can you say about the order of x and y?

The property of order used in these last three examples we call the transitive* property:

> If a, b, c are real numbers
> and if $a < b$ and $b < c$,
> then $a < c$.

Problems

1. In each of the following groups of three real numbers, determine their order:

   For example, $\frac{3}{4}, \frac{3}{2}, {}^-\frac{4}{5}$ have the order: ${}^-\frac{4}{5} < \frac{3}{4}$, $\frac{3}{4} < \frac{3}{2}$, ${}^-\frac{4}{5} < \frac{3}{2}$.

Footnote

*From the Latin, transire, to go across.

(a) $^-\frac{1}{5}$, $\frac{3}{2}$, and 12,

(b) π, $^-\pi$, and $^-\sqrt{2}$,

(c) 1.7, 0, and $^-1.7$,

(d) $^-(\frac{27}{15})$, $^-(\frac{3}{15})$, and $^-(\frac{2}{15})$,

(e) $^-\frac{1}{2}$, $^-\frac{1}{3}$, $^-\frac{1}{4}$,

(f) $1 + \frac{1}{2}$, $1 + (\frac{1}{2})^2$, $(1 + \frac{1}{2})^2$.

2. State a transitive property for ">".

3. Is there a transitive property for the relation "="? If so, give an example.

4. State a transitive property for "$\geq$", and give an example.

5. The set of numbers greater than 0 we have called the *positive real numbers*, and the set of numbers less than 0 the *negative real numbers*. Describe the

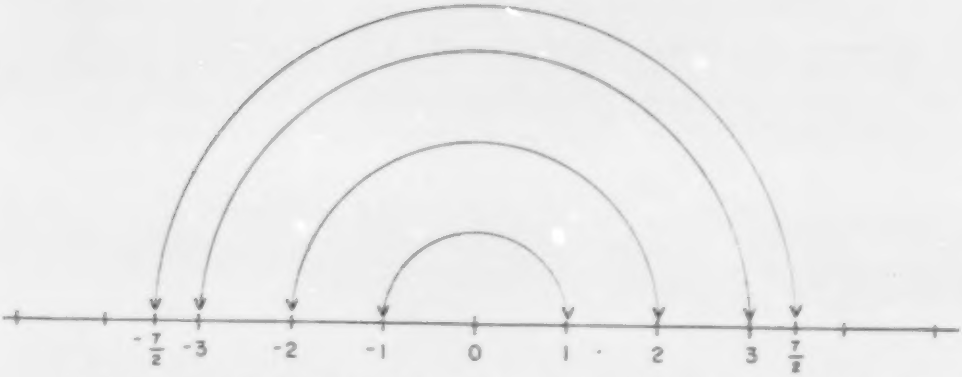
   (a) non-positive real numbers,

   (b) non-negative real numbers.

6. Find the order of each of the following pairs of numbers:

   (a) $^-\frac{145}{28}$ and $^-\frac{104}{21}$ (b) $^-\frac{192}{46}$ and $^-\frac{173}{44}$

Opposites. When we labeled points to the left of 0 on the real number line, we began by marking off successive unit lengths to the left of 0. We can also think, however, of pairing off points at equal distances from 0 and on opposite sides of 0. Thus, $^-2$ is at the same distance from 0 as 2. What number is at the same distance from 0 as $\frac{7}{2}$? If you choose any point on the number line, can you find a point at the same distance from 0 and on the opposite side? What about the point 0 itself?



Since the two numbers in such a pair are on opposite sides of 0, it is natural to call them opposites. The opposite of a non-zero real number is the other real number which is at an equal distance from 0 on the real number line. What is the opposite of 0?

Let us consider some typical real numbers. Write them in a column. Then write their opposites in another column; then study the adjacent statements.

$2, \quad {}^{-}2; \quad {}^{-}2$ is the opposite of 2.

${}^{-}\frac{1}{2}, \quad \frac{1}{2}; \quad \frac{1}{2}$ is the opposite of ${}^{-}\frac{1}{2}$.

$0, \quad 0; \quad 0$ is the opposite of 0.

The statements themselves are cumbersome to write, and we need a symbol meaning "the opposite of". Let us use the lower dash "-" to mean "the opposite of". With this symbol the three statements become the true sentences:

$${}^{-}2 = -2$$

$$\frac{1}{2} = -{}^{-}\frac{1}{2}$$

$$0 = -0.$$

(Read these sentences carefully.)

We can learn two things from these sentences. First, it appears that "$^-2$" and "-2" are different names for the same number. That is, "negative 2" and "the opposite of 2" represent the same number. Hence, it makes no difference at what height the dash is drawn, since the meaning is the same for the upper and lower dash. This being the case, we do not need both symbols.

Which shall we retain? The upper dash refers only to negative numbers, whereas the lower dash may apply to any real number. (Note that the opposite of the positive number 2 is the negative number $^-2$, and the opposite of the negative number $^-\frac{1}{2}$ is the positive number $\frac{1}{2}$.) Hence, it is natural to retain the "opposite of" symbol to mean either "negative" or "opposite of" when the number in question is positive. Now the sentences may be written

$-2 = -2$, (read "negative 2 is the opposite of 2")

$\frac{1}{2} = -(-\frac{1}{2})$, (read "$\frac{1}{2}$ is the opposite of negative $\frac{1}{2}$")

$0 = -0$.

The second of these sentences can be read also as:

$\frac{1}{2}$ is the opposite of the opposite of $\frac{1}{2}$.

Second, we observe in general that the opposite of the opposite of a number is the number itself; in symbols:

For every real number y,

$$-(-y) = y.$$

What is the opposite of the opposite of the opposite of a number? What is the opposite of the opposite of a negative number?

When we attach the dash "-" to a variable x we are performing on x the operation of "determining the opposite of x". Do not confuse this with the binary operation of subtraction, which is performed on two numbers, such as $3 - x$, meaning "x subtracted from 3". What kind of number is $-x$ if x is a positive number? If x is a negative number? If x is 0?

We shall read "-x" as the "opposite of x". Thus, if x is a number to the right of 0 (positive), then -x is to the left (negative); if x is to the left of 0 (negative), then -x is to the right (positive).

Problems

1. Form the opposite of each of the following numbers:

(a)	2.3	(d)	-(3.6 - 2.4)
(b)	-2.3	(e)	-(42 × 0)
(c)	-(-2.3)	(f)	-(42 + 0)

2. What kind of number is -x if x is positive? If x is negative? If x is zero?

3. What kind of number is x if -x is a positive number? If -x is a negative number? If -x is 0?

4. (a) Is every real number the opposite of some real number?
   (b) Is the set of all opposites of real numbers the same as the set of all real numbers?
   (c) Is the set of all negative numbers a subset of the set of all opposites of real numbers?
   (d) Is the set of all opposites of real numbers a subset of the set of all negative numbers?
   (e) Is every opposite of a number a negative number?

The ordering of numbers on the real number line specifies that $-\frac{1}{2}$ is less than 2. Is the opposite of $-\frac{1}{2}$ less than the opposite of 2? Make up other similar examples of pairs of numbers. After you have determined the ordering of a pair, then find the ordering of their opposites. You will see that there is a general property for opposites:

For real numbers a and b,
if $a < b$, then $-b < -a$.

Problems

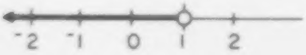
1. Write true sentences for the following numbers and their opposites, using the relations "$<$" or "$>$".
   Example: For the numbers 2 and 7, $2 < 7$ and $-2 > -7$.
   (a) $\frac{2}{7}, -\frac{1}{6}$
   (b) $\sqrt{2}, -\pi$
   (c) $\pi, \frac{22}{7}$
   (d) $3(\frac{4}{3} + 2), \frac{5}{4}(20 + 8)$
   (e) $-(\frac{8 + 6}{7}), -2$

2. Graph the truth sets of the following open sentences:
   (Hint: Use order property of opposites before graphing.)
   (a) $-x > 3$
   (b) $-x > -3$

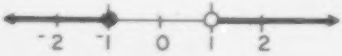
3. Describe the truth set of each open sentence:
   (a) $-x \neq 3$
   (b) $-x \neq -3$
   (c) $-x < 0$
   (d) $-x \leq 0$

4. Write an open sentence for each of the following graphs:

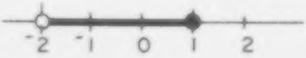
(a)



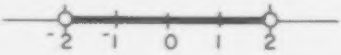
(c)



(b)



(d)



5. For each of the following numbers write its opposite, and then choose the greater of the number and its opposite:

   (a) 0

   (b) 17

   (c) -7.2

   (d) $-\sqrt{2}$

   (e) -0.01

   (f) $-(-2)$

   (g) $-(\frac{1}{2} - \frac{1}{3})$

6. Let us write "$\succ$" for the phrase "is further from 0 than" on the real number line. Does "$\succ$" have the comparison property enjoyed by "$>$", that is, if a and b are different real numbers, is it true that $a \succ b$ or $b \succ a$ but not both? Does "$\succ$" have a transitive property? For which subset of the set of real numbers do "$\succ$" and "$>$" have the same meaning?

7. Change the numerals "$-\frac{13}{42}$" and "$-\frac{15}{49}$" to forms with the same denominators. (Hint: First do this for $\frac{13}{42}$ and $\frac{15}{49}$.) What is the order of $-\frac{13}{42}$ and $-\frac{15}{49}$? (Hint: Knowing the order of $\frac{13}{42}$ and $\frac{15}{49}$, what is the order of their opposites?) Now state a general rule for determining the order of two negative rational numbers.

Absolute Value. We now want to define a new and very useful operation on a single real number: the operation of taking its absolute value.

> The _absolute value_ of a non-zero real number is the greater of that number and its opposite. The absolute value of 0 is 0.

The absolute value of 4 is 4, because the greater of 4 and -4 is 4. The absolute value of $-\frac{3}{2}$ is $\frac{3}{2}$. (Why?) What is the absolute value of -17? Which is always the greater of a non-zero number and its opposite: the positive or the negative number? Explain why the absolute value of any real number is a positive number or 0.

As usual, we agree on a symbol to indicate the operation. We write

$$|n|$$

to mean the absolute value of the number n. For example,

$$|4| = 4, \quad |-\frac{3}{2}| = \frac{3}{2}, \quad |-\sqrt{2}| = \sqrt{2}, \quad |12| = 12.$$

Note that each of these is non-negative.

If you look at these numbers and their absolute values on the number line, what can you conclude about the distance between a number and 0? You notice that the distance between 4 and 0 is 4; between $-\frac{3}{2}$ and 0 is $\frac{3}{2}$, etc. Notice that the distance between any two points of the number line is a non-negative real number.

> The distance between a real
> number and 0 on the real
> number line is the absolute
> value of that number.

We may rephrase our definition of absolute value in a somewhat more formal manner. Note how the use of symbols reduces the amount of language necessary for expressing the idea:

For any real number x
if $x \geq 0$, then $|x| = x$;
if $x < 0$, then $|x| = -x$.

You should verify the truth of the above statements to make sure this really says the same thing as our previous definitions of absolute value.

Problems

1. Find the absolute values of the following numbers:
   - (a) -7
   - (b) $-(-3)$
   - (c) $(6 - 4)$
   - (d) 14×0
   - (e) $-(14 + 0)$
   - (f) $-\big(-(-3)\big)$
2. (a) What kind of number is $\frac{4}{3}$; what kind of number is $|\frac{4}{3}|$? (Non-negative or negative?)

   (b) If x is a non-negative real number, what kind of number is $|x|$?
3. (a) What kind of number is $-\frac{2}{5}$; what kind of number is $|-\frac{2}{5}|$? (Non-negative or negative?)

   (b) If x is a negative real number, what kind of number is $|x|$?
4. Is $|x|$ a non-negative number for every x? Explain.
5. For a negative number x, which is greater, x or $|x|$?
6. Is the set $\{-1, -2, 1, 2\}$ closed under the operation of taking absolute values of its elements?

Problems

1. Which of the following sentences are true?
   - (a) $|-7| < 3$
   - (b) $|-2| \leq |-3|$
   - (c) $|4| < |1|$
   - (d) $2 \nless |-3|$
   - (e) $|-5| \nless |2|$
   - (f) $-3 < 17$
   - (g) $-2 < |-3|$
   - (h) $|\sqrt{16}| > |-4|$
   - (i) $|-2|^2 = 4$

2. Write each as a common numeral:

(a) $|2| + |3|$

(b) $|-2| + |3|$

(c) $-(|2| + |3|)$

(d) $-(|-2| + |3|)$

(e) $|-7| - (7 - 5)$

(f) $7 - |-3|$

(g) $|-5| \times 2$

(h) $-(|-5| - 2)$

(i) $|-3| - |2|$

(j) $|-2| + |-3|$

(k) $-(|-3| - 2)$

(l) $-(|-2| + |-3|)$

(m) $3 - |3 - 2|$

(n) $-(|-7| - 6)$

(o) $|-5| \times |-2|$

(p) $-(|-2| \times 5)$

(q) $-(|-5| \times |-2|)$

3. What is the truth set of each open sentence?

(a) $|x| = 1$

(b) $|x| = 3$

(c) $|x| + 1 = 4$

(d) $5 - |x| = 2$

4. Which of the following open sentences are true for all real numbers x?

(a) $|x| \geq 0$

(b) $x \leq |x|$

(c) $-x < |x|$

(d) $-|x| \leq x$.

(Hint: Give x a positive value; then give x a negative value. Now come to a decision.)

5. Graph the truth sets of the following sentences:

(a) $|x| = 5$

(b) $|x| < 2$

(c) $x > -2$ and $x < 2$

(d) $|x| > 2$

(e) $x < -2$ or $x > 2$

(f) $|x| = -3$ (Be careful.)

6. Graph the set of integers less than 5 whose absolute values are greater than 2. Is -5 an element of this set? Is 0 an element of this set? Is -10 an element of this set?

7. If R is the set of all real numbers, P the set of all positive real numbers, and I the set of all integers, describe the numbers:

   (a) in P but not in I,

   (b) in R but not in P,

   (c) in R but not in P nor in I,

   (d) in P but not in R.

Summary.

1. Points to the left of 0 on the number line are labeled with negative numbers; the set of real numbers consists of all numbers of arithmetic and their opposites.

2. Many points on the number line are not assigned rational number coordinates. These points are labeled with irrational numbers. The set of real numbers consists of all rational and irrational numbers.

3. "Is less than" for real numbers means "to the left of" on the number line.

4. Comparison Property. If a is a real number and b is a real number, then exactly one of the following is true: $a < b$, $a = b$, $b < a$.

5. Transitive Property. If a, b, c are real numbers and if $a < b$ and $b < c$, then $a < c$.

6. The opposite of 0 is 0 and the opposite of any other real number is the other number which is at an equal distance from 0 on the real number line.

7. The absolute value of 0 is 0, and the absolute value of any other real number n is the greater of n and the opposite of n.

8. If x is a positive number, then $-x$ is a negative number. If x is a negative number, then $-x$ is a positive number.

9. The absolute value of the real number n is denoted by $|n|$. Also, $|n|$ is a non-negative number which is the distance between 0 and n on the number line.

10. If $n \geq 0$, then $|n| = n$;
    if $n < 0$, then $|n| = -n$.

Review Problems

1. Consider the open sentence "$|x| < 3$". Draw the graph of its truth set if the domain of x is the set of:

   (a) real numbers

   (b) integers

   (c) non-negative real numbers

   (d) negative integers

2. If R is the set of all real numbers, P the set of all positive real numbers, F the set of all rational numbers, I the set of all integers, which of the following are true statements?

   (a) F is a subset of R.

   (b) Every element of I is an element of F.

   (c) There are elements of I which are not elements of R.

   (d) Every element of I is an element of P.

   (e) There are elements of R which are not elements of F.

3. Draw the graph of the set of integers less than 6 whose absolute values are greater than 3. Is -8 an element of this set?

4. The perimeter of a square is less than 10 inches.

   (a) What do you know about the number of units, s, in the side of this square. Graph this set.

   (b) What do you know about the number of units, A, in the area of this square. Graph this set.

PROPERTIES OF ADDITION

One of the purposes of this section will be to learn how to translate into the language of algebra operations which we first describe geometrically. Addition on the number line is such an operation; we shall try to define it in the language of algebra.

If we take $7 + 5$ and picture this addition on the number line, we first go from 0 to 7, and then from 7 we move 5 more units to the right. If we consider $(-7) + 4$, we first go from 0 to (-7), and then from (-7) move 4 units to the right. These examples remind us of something we already know: To add a positive number, we move to the right on the number line. It should now be clear from our other examples above what happens on the number line when we add a negative number. When we added (-4), we moved 4 units to the left; when we added (-6), we moved 6 units to the left. We have one more case to consider: If we add 0, what motion, if any, results?

We have now described the motion in all cases; let us see if we can learn to say algebraically how far we move. Forget for the moment the direction; we just want to know how far we go when we go from a to $a + b$. When b is positive we go to the right. Yes, but how far? We go just b units. When b is negative, we go to the left. How far? We go $(-b)$ units. (Remember $(-b)$ is positive if b is negative.) If b is 0, we don't go at all. What symbol do we know which means "b if b is positive, $-b$ if b is negative, and 0 if b is 0"? "$|b|$", of course. And so we have learned that to find $a + b$ on the number line, we start from a and move the distance $|b|$

to the right, if b is positive;
to the left, if b is negative.

Problems

1. Perform the indicated operations on real numbers, using the number line to aid you:
   - (a) $\Big(4 + (-6)\Big) + (-4)$
   - (b) $4 + \Big((-6) + (-4)\Big)$
   - (c) $-\Big(4 + (-6)\Big)$
   - (d) $3 + \Big((-2) + 2\Big)$
   - (e) $2 + \Big(0 + (-2)\Big)$
   - (f) $\Big((-3) + 0\Big) + (-2.5)$
   - (g) $|-2| + (-2)$
   - (h) $(-3) + (|-3| + 5)$
2. Tell in your own words what you do to the two given numbers to find their sum:
   - (a) $7 + 10$
   - (b) $7 + (-10)$
   - (c) $10 + (-7)$
   - (d) $(-10) + (-7)$
   - (e) $10 + 7$
   - (f) $(-7) + (-10)$
   - (g) $(-7) + 10$
   - (h) $(-10) + 7$
   - (i) $(-10) + 0$
   - (j) $0 + 7$
3. In which parts of Problem 2 did you do the addition just as you added numbers in arithmetic?
4. What could you always say about the sum when both numbers were negative?

Properties of Addition. One of our main objectives in this course is to study the structure of the real number system. A *system* of numbers is a set of numbers and the operations on these numbers. Hence, we do not really have the real number *system* until we *define* the operations of addition and multiplication for negative numbers.

The operations must be extended from the non-negative reals to all real numbers. Thus, the definitions of addition and multiplication for all real numbers must be formulated exclusively in terms of the non-negative numbers and operations (including oppositing) on them. We, of course, insist on preserving the fundamental properties of the operations.

In arithmetic, the sign "+" in the expression "25 + 38" is nothing more than a reminder or command to carry out a previously learned process to obtain "63". The idea of "+" as an operation to be studied for its own sake is quite a different notion of addition from that in arithmetic.

An attempt is made here to begin thinking of the real number system from the deductive point of view. In other words, it is an undefined set of elements endowed with an operation of addition, an operation of multiplication and an order relation subject to certain assumed properties from which all other properties can be deduced by proofs.

Very quickly in the present section you should learn how to find sums involving negative numbers. We want to bring out the important fact that what is really involved here is an extension of the operation of addition from the numbers of arithmetic (where the operation is familiar) to all real numbers in such a way that the basic properties of addition are preserved. This means that we must give a definition of addition in terms of only non-negative numbers and familiar operations on them. The result in the language of algebra is a formula for $a + b$ involving the familiar operations of addition, subtraction and opposite applied to the non-negative $|a|$ and $|b|$. The complete formula appears formidable because of the variety of cases. However, the idea is simple and is nothing more than a general statement of exactly what we always do in obtaining the sum of negative numbers.

Definition of Addition. We now want to use what we have just learned about addition on the number line to say, first in English and then in the language of algebra, what we mean by $a + b$ for all real numbers a and b. First of all, we know from previous experience how to add a and b if both are non-negative numbers. So let us consider another example, namely, a negative plus a negative. What is

$$(-4) + (-6)?$$

We have found, on the number line, that

$$(-4) + (-6) = (-10).$$

Our present job is to think a bit more carefully about just how we reached (-10). We begin by moving from 0 to (-4). Where is (-4) on the number line? It is to the left of 0. How far? "Distance between a number and 0" was one of the meanings of the absolute value of a number. Thus, the distance between 0 and (-4) is $|-4|$. (Of course, we realize that it is easier to write 4 than $|-4|$, but the expression $|-4|$ reminds us that we were thinking of "distance from 0", and this is worth remembering at present.)

(-4) is thus $|-4|$ to the left of 0. When we now consider

$$(-4) + (-6),$$

we move another $|-6|$ to the left. Where are we now? At

$$-(|-4| + |-6|).$$

Thus, our thinking about distance from 0 and about distance moved on the number line has led us to recognize that

$$(-4) + (-6) = -(|-4| + |-6|)$$

is a true sentence.

You can reasonably ask at this point what we have accomplished by all this. We have taken a simple expression like $(-4) + (-6)$, and made it look more complicated! Yes, but the expression $-(|-4| + |-6|)$, complicated as it looks, has one great advantage.

It contains only operations which we know how to do from previous experience! Both $|-4|$ and $|-6|$ are positive numbers, you see, and we know how to add positive numbers; and $-(|-4| + |-6|)$ is the opposite of a number, and we know how to find that. Thus, we *have* succeeded in expressing the sum of two negative numbers for which sum we previously had just a picture on the number line, in terms of the language of algebra as we have built it up thus far.

Think through $(-2) + (-3)$ for yourself, and see that by the same reasoning you arrive at the true sentence

$$(-2) + (-3) = -(|-2| + |-3|).$$

From these examples we see that the following defines the sum of two negative numbers in terms of operations which we already know how to do:

In English: The sum of two negative numbers is negative; the absolute value of this sum is the sum of the absolute values of the numbers.

In the language of algebra:

If a and b are both negative numbers, then

$$a + b = -(|a| + |b|).$$

Problems

1. Use the definition above to find a common name for each of the following indicated sums, and then check by using the number line. Example: by definition

$$\begin{aligned}(-2) + (-3) &= -(|-2| + |-3|)\\ &= -(2 + 3)\\ &= -5\end{aligned}$$

Check: A loss of $2 followed by a loss of $3 is a net loss of $5.

(a) $(-2) + (-7)$

(b) $(-4.6) + (-1.6)$

(c) $(-3\frac{1}{3}) + (-2\frac{2}{3})$

(d) $(-25) + (-73)$

(e) $5\frac{1}{2} + 2\frac{1}{2}$

2. Find a common name for each of the following by any method you choose:

(a) $(-6) + (-7)$

(b) $(-7) + (-6)$

(c) $-(|-7| + |-6|)$

(d) $6 + (-4)$

(e) $(-4) + 6$

(f) $|6| - |-4|$

(g) $0 + (-3)$

(h) $-(|-3| - |0|)$

(i) $3 + \big((-2) + 2\big)$

3. When one number is positive and the other is negative, how do you know whether the sum is positive or negative?

So far, we have considered the sum of two non-negative numbers, and the sum of two negative numbers. Next we consider the sum of two numbers, of which one is positive and the other is negative.

Let us look at a few examples of gains and losses:

Profit of $7 and loss of $3; $7 + (-3) = 4$; $|7| - |-3| = |4|$

Profit of $3 and loss of $7; $3 + (-7) = -4$; $|-7| - |3| = |-4|$

Loss of $7 and profit of $3; $(-7) + 3 = -4$; $|-7| - |3| = |-4|$

Loss of $3 and profit of $7; $(-3) + 7 = 4$; $|7| - |-3| = |4|$

Loss of $3 and profit of $3; $(-3) + 3 = 0$; $|3| - |-3| = |0|$

Consider these examples on the number line. From these it appears that the sum of two numbers, of which one is positive (or 0) and the other is negative, is obtained as follows:

The absolute value of the sum is the difference of the absolute values of the numbers.

The sum is positive if the positive number has the greater absolute value.

The sum is negative if the negative number has the greater absolute value.

The sum is 0 if the positive and negative numbers have the same absolute value.

In the language of algebra,

If $a \geq 0$ and $b < 0$, then:

$$a + b = |a| - |b|, \text{ if } |a| \geq |b|$$

and

$$a + b = -(|b| - |a|), \text{ if } |b| > |a|.$$

If $b \geq 0$ and $a < 0$, then:

$$a + b = |b| - |a|, \text{ if } |b| \geq |a|$$

and

$$a + b = -(|a| - |b|), \text{ if } |a| > |b|.$$

Problems

1. In each of the following, find the sum, first according to the definition, and then by any other method you find convenient.

(a) $(-5) + 3$	(e) $18 + (-14)$
(b) $(-11) + (-5)$	(f) $12 + 7.4$
(c) $(-\frac{8}{3}) + 0$	(g) $(-\frac{2}{3}) + 5$
(d) $2 + (-2)$	(h) $(-35) + (-65)$

2. Is the set of all real numbers closed under the operation of addition?

3. Is the set of all negative real numbers closed under addition? Justify your answer.

4. For each of the following open sentences, find a real number which will make the sentence true:

(a) $x + 2 = 7$

(b) $3 + y = -7$

(c) $a + 5 = 0$

(d) $b + (-7) = 3$

(e) $(-\frac{5}{6}) + x = -\frac{5}{6}$

(f) $c + (-3) = -7$

(g) $y + \frac{2}{3} = -\frac{5}{6}$

(h) $\frac{1}{2}x + (-4) = 6$

(i) $\left(y + (-2)\right) + 2 = 3$

(j) $(3 + x) + (-3) = -1$

5. Which of the following sentences are true?

(a) $-(|-1.5| - |0|) = -1.5$

(b) $(-3) + 5 = 5 + (-3)$

(c) $\left(4 + (-6)\right) + 6 = 4 + \left((-6) + 6\right)$

(d) $(-5) + \left(-(-5)\right) = -10$

(e) $-\left(6 + (-2)\right) = (-6) + (-2)$

6. Translate the following English sentences into open sentences. For example: Bill spent 60¢ on Tuesday and earned 40¢ on Wednesday. He couldn't remember what happened on Monday, but he had 30¢ left on Wednesday night. What amount did he have on Monday?

If Bill had x cents on Monday, then

$$x + (-60) + 40 = 30.$$

This can be written

$$x + (-20) = 30.$$

(a) If you drive 40 miles north and then drive 55 miles south, how far are you from your starting point?

(b) The sum of (-9), 28, and a third number is (-52). What is the third number?

(c) At 8 A.M. the temperature was -2°. Between 8 A.M. and noon the temperature increased 15°. Between noon and 4 P.M. the temperature increased 6°. At 8 P.M. the temperature was -9°. What was the temperature change between 4 P.M. and 8 P.M.?

(d) If a 200-pound man lost 4 pounds one week, lost 6 pounds the second week, and at the end of the third week weighed 195 pounds, how much did he gain in the third week?

(e) A stock which was listed at 83 at closing time Monday dropped 5 points on Tuesday. Thursday morning it was listed at 86. What was the change on Wednesday?

Properties of Addition. We were careful to describe and list the properties of addition when we dealt with the numbers of arithmetic. Now that we have decided how to add real numbers, we want to verify that these properties of addition hold true for the real numbers generally.

We know that our definition of addition includes the usual addition of numbers of arithmetic, but we also want to be able to add as simply as we could before. Can we still add real numbers in any order and group them in any way to suit our convenience? In other words, do the commutative and associative properties of addition still hold true? If we are able to satisfy ourselves that these properties do carry over to the real numbers, then we are assured that the structure of numbers is maintained as we move from the numbers of arithmetic to the real numbers. Similar questions about multiplication will come up later.

Consider the following question: Are $4 + (-3)$ and $(-3) + 4$ names for the same number?

It appears that the sum of any two real numbers is the same for either order of addition. This is the

Commutative Property of Addition: For any two real numbers a and b,

$$a + b = b + a.$$

Next, compute the following pair of sums:

$$\big(7 + (-9)\big) + 3, \quad \text{and} \quad 7 + \big((-9) + 3\big);$$

What do you observe about the results?

We could list many more examples. Do you think the same results would always hold? We have the

Associative Property of Addition: For any real numbers a, b, and c,

$$(a + b) + c = a + (b + c).$$

Of course, if the associative and commutative properties hold true in several instances it is not a proof that they will hold true in every instance. A complete proof of the properties can be given by applying the precise definition of addition of real numbers to every possible case of the properties. They are long proofs, especially of the associative property, because there are many cases. We shall not take the time to give the proofs, but perhaps you may want to try the proof for the commutative property in some of the cases.

The associative property assures us that in a sum of three real numbers it doesn't matter which adjacent pair we add first; it is customary to drop the parenthese and leave such sums in an unspecified form, such as $4 + (-1) + 3$.

Another property of addition, which is new for real numbers and one that we shall find useful, is obtained from the definition of addition. For example, the definition tells us that $4 + (-4) = 0$; that $(-4) + \big(-(-4)\big) = 0$. In general, the sum of a number and its opposite is 0. We state this as the

Addition Property of Opposites: For every real number a,

$$a + (-a) = 0.$$

One more property that stems directly from the definition is the

Addition Property of 0: For every real number a,

$$a + 0 = a.$$

Problems

1. Show how the properties of addition can be used to explain why each of the following sentences is true:

Example:

$$5 + \left(3 + (-5)\right) = 3 + 0.$$

The left numeral is

$$5 + \left(3 + (-5)\right) = \left(5 + (-5)\right) + 3$$ associative and commutative properties of addition.

$$= 0 + 3$$ addition property of opposites.

$$= 3 + 0$$ commutative property of addition.

The right numeral is

$$3 + 0.$$

(a) $3 + \left((-3) + 4\right) = 0 + 4$

(b) $\left(5 + (-3)\right) + 7 = \left((-3) + 5\right) + 7$

(c) $\left(7 + (-7)\right) + 6 = 6$

(d) $|-1| + |-3| + (-3) = 1$

(e) $(-2) + \left(3 + (-4)\right) = \left((-2) + 3\right) + (-4)$

(f) $(-|-5|) + 6 = 6 + (-5)$

2. Using the associative and commutative properties of addition, write a simpler name for one phrase of each of the following sentences, and find the truth set of each:

   (a) $x = x + \big((-x) + 3\big)$

   (b) $m + \big(7 + (-m)\big) = m$

   (c) $n + (n + 2) + (-n) + 1 + (-3) = 0$

   (d) $(y + 4) + (-4) = 9 + (-4)$

The Addition Property of Equality. There is another fact about addition to which we must give attention. We know that

$$4 + (-5) = (-1).$$

This means that $4 + (-5)$ and (-1) are two names for one number. Let us add 3 to that number. Then $\big(4 + (-5)\big) + 3$ and $(-1) + 3$ are again two names for one number. Thus,

$$\big(4 + (-5)\big) + 3 = (-1) + 3.$$

Also, for example,

$$\big(4 + (-5)\big) + 5 = (-1) + 5.$$

Similarly, since

$$7 = 15 + (-8),$$
$$7 + (-7) = \big(15 + (-8)\big) + (-7).$$

This suggests the

Addition Property of Equality: For any real numbers $a, b, c,$

$$\text{if } a = b, \text{ then } a + c = b + c.$$

In words, if a and b are two names for one number, then $a + c$ and $b + c$ are two names for one number.

Let us use the previously stated properties of addition, and the above property of equality in some examples.

Example 1. Determine the truth set of the open sentence

$$x + \frac{3}{5} = -2.$$

Can you guess numbers which make this sentence true? If you don't see it easily, could you use properties of addition to help? Let us see. We do not really know whether there is any number making this sentence true. If, however, there is such a number x which makes the sentence true (that is, if the truth set is not empty), then $x + \frac{3}{5}$ and -2 are the same number.

Let us add $-\frac{3}{5}$ to this number; then by the addition property of equality we have

$$\left(x + \frac{3}{5}\right) + \left(-\frac{3}{5}\right) = (-2) + \left(-\frac{3}{5}\right).$$

Why did we add $-\frac{3}{5}$? Because in this case we wish to change the left numeral so it will contain the numeral "x" alone. Watch this happening in the next few lines.

Continuing, we have

$$x + \left(\frac{3}{5} + \left(-\frac{3}{5}\right)\right) = (-2) + \left(-\frac{3}{5}\right). \qquad \text{(Why?)}$$

$$x + 0 = -\frac{13}{5}. \qquad \text{(Why?)}$$

$$x = -\frac{13}{5}. \qquad \text{(Why?)}$$

Thus, we arrive at the new open sentence $x = -\frac{13}{5}$. If a number x makes the original sentence true, it also makes this new sentence true. Of this we are certain because we applied properties which hold true for all real numbers. This tells us that $-\frac{13}{5}$ is the only possible truth value of the original sentence. But it does not guarentee that it is a truth value. Does $-\frac{13}{5}$ make the original sentence true? Yes, because $\left(-\frac{13}{5}\right) + \frac{3}{5} = -2$.

Here we have discovered a very important idea about sentences such as the above. We have shown that if there is a number x making the original sentence true, then the only number which x can be is $-\frac{13}{5}$. The minute we check and find that $-\frac{13}{5}$ does make the sentence true, we have found the one and only number which belongs to the truth set.

The sentence in the previous example is an equation. We shall often call the truth set of an equation its solution set, and its members solutions, and we shall write "solve" instead of "determine the truth set of".

Example 2. Solve the equation

$$5 + \frac{3}{2} = x + (-\frac{1}{2}).$$

If $5 + \frac{3}{2} = x + (-\frac{1}{2})$ is true for some x,

then $(5 + \frac{3}{2}) + \frac{1}{2} = \left(x + (-\frac{1}{2})\right) + \frac{1}{2}$ is true for the same x;

$5 + 2 = x + 0$ is true for the same x;

$7 = x$ is true for the same x.

If $x = 7$,

the left side is:

$$5 + \frac{3}{2} = \frac{10}{2} + \frac{3}{2} = \frac{13}{2}$$

the right side is:

$$7 + (-\frac{1}{2}) = \frac{14}{2} + (-\frac{1}{2}) = \frac{13}{2}$$

Hence, the truth set is $\{7\}$.

Problems

Solve each of the following equations. Write your work in the form shown for Example 2 above.

1. $x + 5 = 13$
2. $(-6) + 7 = (-8) + x$
3. $(-1) + 2 + (-3) = 4 + x + (-5)$
4. $(x + 2) + x = (-3) + x$
5. $(-2) + x + (-3) = x + (-\frac{5}{2})$
6. $|x| + (-3) = |-2| + 5$
7. $(-\frac{3}{8}) + |x| = (-\frac{3}{4}) + (-1)$
8. $x + (-3) = |-4| + (-3)$
9. $(-\frac{4}{3}) + (x + \frac{1}{2}) = x + (x + \frac{1}{2})$

The Additive Inverse. Two numbers whose sum is 0 are related in a very special way. For example, what number when added to 3 yields the sum 0? What number when added to -4 yields 0? In general, if x and y are real numbers such that

$$x + y = 0,$$

we say that y is an additive inverse of x. Under this definition, is x then also an additive inverse of y?

Now let us think about any number z which is an additive inverse of, say, 3. Of course, we know one such number, namely, -3; for by the addition property of opposites, $3 + (-3) = 0$. Can there be any other number z such that

$$3 + z = 0?$$

All of our experience with numbers tells us "No, there is no other such number". But how can we be absolutely sure? We can settle this question with the use of our properties of addition, just as we did in Example 1 in the preceding section. If, for some number z,

$$3 + z = 0$$

is a true sentence, then

$$(-3) + (3 + z) = (-3) + 0$$

is also a true sentence, by the addition property of equality. (Why did we add -3?) Then, however,

$$\big((-3) + 3\big) + z = -3$$

is true for the same x by the associative property of addition and the 0 property of addition. This finally tells us that

$$z = -3$$

must also be true; we have, for this last step, used the addition property of opposites.

What have we done here? We started out by choosing z as *any* number which is an additive inverse of 3; we found out that z *had* to equal -3; that is, that -3 is not just *an* additive inverse of 3, but also the *only* additive inverse of 3.

Is there anything special about 3? Do you think 5 has more than one additive inverse? How about (-6.3)? We certainly doubt it, and we can show that they do not by the same line of reasoning as the above. Can we, however, check *all* numbers? What we need is a result for *any* real number x, a result which is supposed to tell us something like the following: We know that (-x) is *one* additive inverse of x; we doubt if there is any other, and this is how we prove there is none. Let us parallel the reasoning we used in the special case in which x = 3, and see if we can arrive at the corresponding conclusion.

Suppose z is any additive inverse of x, that is, any number such that

$$x + z = 0.$$

What corresponds to the first step in our previous special case? We use the addition property of equality to write

$$(-x) + (x + z) = (-x) + 0.$$

We then have that

$$\big((-x) + x\big) + z = -x.$$

What are the two reasons we have used in arriving at this last sentence?

Then

$$0 + z = -x, \qquad \text{(Why?)}$$

and finally

$$z = -x. \qquad \text{(Why?)}$$

We have succeeded in carrying out our program, not just when $x = 3$ but for any x. Each number x has a *unique* (meaning "just one") additive inverse, namely, $-x$.

You probably have all kinds of qualms and questions at this point, and these are to be expected since this is the first *proof* which you have seen in this course. What we have done is to use facts which we have *previously* known about all real numbers in order to argue out a *new* fact about all real numbers, a new fact which you certainly expected to be true, but which nevertheless took this kind of checking. We shall do a number of proofs in this course, and you will become more and more accustomed to this kind of reasoning as you progress. In the meantime, let us make one more comment about the proof just completed. The successive steps we took were of course chosen quite deliberately in order to make the proof succeed. This might give you the impression that the proof was "rigged", that it couldn't come out any other way. Is this fair? Yes, it is, and in fact every proof is "rigged" in the sense that we take only steps to help us towards our goal, and do not take steps which fail to do us any good. When we started from

$$3 + z = 0,$$

we chose to use the addition property of equality to add (-3); we could have added any other number instead, but it wouldn't have helped us. And so we didn't add a different number, but added -3.

Statements of new facts or properties, which can be shown to follow from previously established properties, are frequently (but not always!) called "theorems". Thus, the property about additive inverses obtained above can be stated as a theorem:

> Theorem. Any real number x has exactly one additive inverse, namely, $-x$.

An argument by which a theorem is shown to be a consequence of other properties is called a proof of the theorem.

Problems

1. For each sentence, find its truth set.

(a) $3 + x = 0$

(b) $(-2) + a = 0$

(c) $3 + 5 + y = 0$

(d) $x + (-\frac{1}{2}) = 0$

(e) $|-4| + 3 + (-4) + c = 0$

(f) $\left(-(-\frac{2}{3})\right) + y = 0$

(g) $\left(-(2 + \frac{1}{3})\right) + a = 0$

(h) $2 + x + (-5) = 0$

(i) $3 + (-x) = 0$

2. Were you able to use the above theorem to save work in solving these equations?

Let us look at another example for this technique of showing a general property of numbers. Of course, we cannot prove a general property of numbers until we suspect one; let us find one to suspect. Recall the picture of addition on the number line, or the definition of addition if you prefer, to see that

$$(-3) + (-5) = -(3 + 5).$$

Another way of writing that $(-3) + (-5)$ and $-(3 + 5)$ are names for the same number is that

$$-(3 + 5) = (-3) + (-5).$$

This might lead us to suspect that the opposite of the sum of two numbers is the sum of the opposites. Of course, we have checked this only for the numbers 3 and 5, and it is wise to check a few more cases. Is

$$-(2 + 9) = (-2) + (-9)?$$

Is

$$-\big(4 + (-2)\big) = (-4) + \big(-(-2)\big)?$$

(What is another name for $\big(-(-2)\big)$?)
Is

$$-\big((-1) + (-4)\big) = 1 + 4?$$

Our hunch seems to be true, at least in all the examples we have tried. Let us now, instead of checking any more examples by arithmetic, state the general property which we hope to prove as a theorem.

Theorem. For any real numbers a and b

$$-(a + b) = (-a) + (-b).$$

Proof. We need to prove that $(-a) + (-b)$ names the same number as $-(a + b)$. Let us check that $(-a) + (-b)$ acts like the opposite of $(a + b)$. We look at $(a + b) + \big((-a) + (-b)\big)$, for if this expression is 0, $(-a) + (-b)$ will be the opposite of $(a + b)$.

$$\begin{aligned}
(a + b) + \big((-a) + (-b)\big) &= a + b + (-a) + (-b) \\
&= \big(a + (-a)\big) + \big(b + (-b)\big) \qquad \text{(Why?)} \\
&= 0 + 0 \qquad \text{(Why?)} \\
&= 0.
\end{aligned}$$

And so we find that for all real numbers a and b,

$$(-a) + (-b)$$

is an additive inverse of $(a + b)$, and, since there is only one additive inverse, that

$$-(a + b) \text{ and } (-a) + (-b)$$

name the same number.

Problems

1. Which of the following sentences are true for all real numbers? Hint: Remember that the opposite of the sum of two numbers is the sum of their opposites.

   (a) $-(x + y) = (-x) + (-y)$

   (b) $-x = -(-x)$

   (c) $-(-x) = x$

   (d) $-\big(x + (-2)\big) = (-x) + 2$

   (e) $-\big(a + (-b)\big) = (-a) + b$

   (f) $\big(a + (-b)\big) + (-a) = b$

   (g) $-\big(x + (-x)\big) = x + (-x)$

2. In the following proof supply the reason for each step: For all numbers x, y and z,

$$(-x) + \big(y + (-z)\big) = y + \big(-(x + z)\big).$$

Proof:

$$\begin{aligned}(-x) + \big(y + (-z)\big) &= (-x) + \big((-z) + y\big)\\ &= \big((-x) + (-z)\big) + y\\ &= \big(-(x + z)\big) + y\\ &= y + \big(-(x + z)\big).\end{aligned}$$

3. Is $-\left(3 + 6 + (-4) + 5\right) = (-3) + (-6) + 4 + (-5)$? What do you think is true for the opposite of the sum of more than two numbers?

Tell which of the following sentences are true.

(a) $-\left((-2) + 6 + (-5)\right) = 2 + (-6) + 5$

(b) $-\left(3a + (-b) + (-2)\right) = 3a + b + 2$

(c) $-\left(a + (-b) + (-5c) + .7d\right) = (-a) + b + 5c + (-.7d)$

(d) $-\left(\frac{5}{3}x + 2y + (-2a) + (-3b)\right) = (-\frac{5}{3}x) + 2y + (-2a) + (-3b)$

4. Prove the following property of addition:

For any real number a and any real number b and any real number c,

$$\text{if } a + c = b + c, \text{ then } a = b.$$

Summary.

We have defined addition of real numbers as follows:

The sum of two positive numbers is familiar from arithmetic.

The sum of two negative numbers is negative; the absolute value of this sum is the sum of the absolute values of the numbers.

The sum of two numbers, of which one is positive (or 0) and the other is negative, is obtained as follows:

The absolute value of the sum is the difference of the absolute values of the numbers.

The sum is positive if the positive number has the greater absolute value.

The sum is negative if the negative number has the greater absolute value.

The sum is 0 if the positive and negative numbers have the same absolute value.

We have satisfied ourselves that the following properties hold for addition of real numbers:

Commutative Property of Addition: For any two real numbers a and b,

$$a + b = b + a.$$

Associative Property of Addition: For any real numbers a, b, and c,

$$(a + b) + c = a + (b + c).$$

Addition Property of Opposites: For every real number a,

$$a + (-a) = 0.$$

Addition Property of 0: For every real number a,

$$a + 0 = a.$$

Addition Property of Equality: For any real numbers a, b, and c,

$$\text{if } a = b, \text{ then } a + c = b + c.$$

We have used the addition property of equality to determine the truth sets of open sentences.

We have proved that the additive inverse is unique - that is, that each number has exactly one additive inverse, which we call its opposite.

We have discovered and proved the fact that the opposite of the sum of two numbers is the same as the sum of their opposites?

Review Problems

1. Show how the properties of addition can be used to explain why each of the following sentences is true:

   (a) $\frac{2}{3} + \left(7 + \left(-\frac{2}{3}\right)\right) = 7$

   (b) $|-5| + (-.36) + |-.36| = 10 + \left(2 + (-7)\right)$

2. Find the truth set of each of the following:

   (a) $\frac{5}{9} + 32 = x + \frac{5}{9}$

   (b) $x + 5 + (-x) = 12 + (-x) + (-3)$

   (c) $3x + \frac{15}{2} + x = 10 + 3x + (-\frac{7}{2})$

   (d) $|x| + 3 = 5 + |x|$

3. For what set of numbers is each of the following sentences true?

   (a) $|3| + |a| > |-3|$

   (b) $|3| + |a| = |-3|$

   (c) $|3| + |a| < |-3|$

4. Two numbers are added. What do you know about these numbers if

   (a) their sum is negative?

   (b) their sum is 0?

   (c) their sum is positive?

5. A figure has four sides. Three of them are 8 feet, 10 feet, and 5 feet, respectively. How long is the fourth side?

   (a) Write a compound open sentence for this problem.

   (b) Graph the truth set of the open sentence.

6. If a, b, and c are numbers of arithmetic, write each of the indicated sums as an indicated product, and each of the indicated products as an indicated sum:

   (a) $(2b + c)a$

   (b) $2a(b + c)$

   (c) $3a + 3b$

   (d) $5x + 10ax$

   (e) $x^2y + xy$

   (f) $6a^2b + 2ab^2$

   (g) $ab(ac + 3b)$

   (h) $3a(a + 2b + 3c)$

7. Given the set $\{-5, 0, \frac{3}{4}, -.75, 5\}$:

(a) Is this set closed under the operation of taking the opposite of each element of the set?

(b) Is this set closed under the operation of taking the absolute value of each element?

(c) If a set is closed under the operation of taking the opposite, is it closed under the operation of taking the absolute value? Why?

8. Given the set $\{-5, 0, \frac{3}{4}, 5, 7\}$:

(a) Is this set closed under the operation of taking the absolute value of each element of the set?

(b) Is this set closed under the operation of taking the opposite of each element?

(c) If a set is closed under the operation of taking the absolute value, is it closed under the operation of taking the opposite? Why?

PROPERTIES OF MULTIPLICATION

Multiplication of Real Numbers. Now let us decide how we should multiply two real numbers to obtain another real number. All that we can say at present is that we know how to multiply *two non-negative numbers*.

Of primary importance here, as in the definition of addition, is that we maintain the "structure" of the number system. We know that if a, b, c are any *numbers of arithmetic*, then

$$\begin{gathered} ab = ba, \\ (ab)c = a(bc), \\ a \cdot 1 = a, \\ a \cdot 0 = 0, \\ a(b + c) = ab + ac. \end{gathered}$$

(What names did we give to these properties of multiplication?) Whatever meaning we give to the product of two real numbers, we must be sure that it agrees with the products which we already have for non-negative real numbers and that the above *properties of multiplication still hold for all real numbers*.

Consider some possible products:

$$(2)(3),\ (3)(0),\ (0)(0),\ (-3)(0),\ (3)(-2),\ (-2)(-3).$$

(Do these include examples of every case of multiplication of positive and negative numbers and zero?) Notice that the first three products involve only non-negative numbers and are therefore already determined:

$$(2)(3) = 6,\quad (3)(0) = 0,\quad (0)(0) = (0).$$

Now let us try to see what the reamining three products will have to be in order to preserve the basic properties of multiplication listed above. In the first place, if we want the multiplication property of 0 to hold for all real numbers, then we must have $(-3)(0) = 0$. The other two products can be obtained as follows:

$0 = (3)(0)$

$0 = (3)\big(2 + (-2)\big)$, by writing $0 = 2 + (-2)$; (Notice how this introduces a negative number into the discussion.)

$0 = (3)(2) + (3)(-2)$, if the distributive property is to hold for real numbers;

$0 = 6 + (3)(-2)$, since $(3)(2) = 6$.

We know from uniqueness of the additive inverse that the only real number which yields 0 when added to 6 is the number -6. Therefore, if the properties of numbers are expected to hold, the only possible value for $(3)(-2)$ which we can accept is -6.

Next, we take a similar course to answer the second question.

$0 = (-2)(0)$ if the multiplication property of 0 is to hold for real numbers;

$0 = (-2)\big(3 + (-3)\big)$, by writing $0 = 3 + (-3)$;

$0 = (-2)(3) + (-2)(-3)$, if the distributive property is to hold for real numbers;

$0 = (3)(-2) + (-2)(-3)$, if the commutative property is to hold for real numbers;

$0 = (-6) + (-2)(-3)$, by the previous result, which was $(3)(-2) = -6$.

Now we have to come to a point where $(-2)(-3)$ must be the opposite of -6; hence, if we want the properties of multiplication to hold for real numbers, then $(-2)(-3)$ must be 6.

Let us think of these examples now in terms of absolute value.

Recall that the product of two positive numbers is a positive number. Then what are the values of $|3||2|$ and $|-2||-3|$? How do these compare, respectively, with $(3)(2)$ and $(-2)(-3)$? Compare $(-3)(4)$ and $-(|-3||4|)$; $(-5)(-3)$ and $|-5||-3|$; $(0)(-2)$ and $|0||-2|$.

This is the hint we needed. If we want the structure of the number system to be the same for real numbers as it was for the numbers of arithmetic, we must define the product of two real numbers a and b as follows:

> If a and b are both negative or both non-negative, then $ab = |a||b|$.
>
> If one of the numbers a and b is non-negative and the other is negative, then $ab = -(|a||b|)$.

It is important to recognize that $|a|$ and $|b|$ are numbers of arithmetic for any real numbers a and b; and we already know the product $|a||b|$. (Why?) Thus, the product $|a||b|$ is a positive number or zero, and we obtain the product ab as either $|a||b|$ or its opposite. Again we have used only the operations with which we are already familiar: multiplying positive numbers or 0, and taking opposites.

The product of two positive numbers is a ________ number.
The product of two negative numbers is a ________ number.
The product of a negative and a positive number is a ______ number.
The product of a real number and 0 is _________.

Since the product ab is either $|a||b|$ or its opposite and since $|a||b|$ is non-negative, we can state the following property of multiplication:

For any real numbers a, b,

$$|ab| = |a||b|.$$

Problems

1. Calculate the following:

(a) $\left((-\frac{1}{2})(2)\right)(-5)$

(b) $(-\frac{1}{2})\left((2)(-5)\right)$

(c) $(-3)(-4) + (-3)(7)$

(d) $(-3)\left((-4) + 7\right)$

(e) $(-3)(-4) + 7$

(f) $|-3|(-4) + 7$

(g) $|3||-2| + (-6)$

(h) $(-3)|-2| + (-6)$

(i) $(-3)\left(|-2| + (-6)\right)$

(j) $(-0.5)\left(|-1.5| + (-4.2)\right)$

2. Find the values of the following for $x = -2$, $y = 3$, $a = -4$:

(a) $x^2 + 2(xa) + a^2$

(b) $(x + a)^2$

(c) $x^2 + \left(3|a| + (-4)|y|\right)$

(d) $|x + 2| + (-5)|(-3) + a|$

3. Which of the following sentences are true?

(a) $2(-y) + 8 = 28$, for $y = -10$

(b) $(-5)\left((-b)(-4) + 30\right) < 0$, for $b = 2$

(c) $|x + 3| + (-2)\left(|x + (-4)|\right) \geq 1$, for $x = 2$

4. Find the truth sets of the following open sentences and draw their graphs.

Example: Find the truth set of $(3)(-3) + c = 3(-4)$.

If $(3)(-3) + c = 3(-4)$ is true for some c,

then $-9 + c = -12$ is true for the same c;

$(-9 + c) + 9 = -12 + 9$ is true for the same c;

$c = -3$.

If $c = -3$,

then the left member is $(3)(-3) + (-3) = -12$,

and the right member is $3(-4) = -12$.

Hence, the truth set is $\{3\}$.

(a) $x + (-3)(-4) = 8$

(b) $x + 2 = 3(-6) + (-4)(-8)$

(c) $x = (-5)(-6) + |-2|(3)$

(d) $x > (-4)(-2) + (-5)(2)$

(e) $|x| = (-\frac{2}{3})(7) + (-1)(-5)$

5. Given the set R of all real numbers, find the set Q of all products of pairs of elements of R. Is Q the same set as R? Can you conclude that R is closed under multiplication?

6. Given the set $V = \{1, -2, -3, 4\}$, find the set K of all positive numbers obtained as products of pairs of elements of V.

7. Prove that the absolute value of the product ab is the product $|a|\cdot|b|$ of the absolute values; that is,

$$|ab| = |a||b|.$$

8. What can you say about two real numbers a and b in each of the following cases?

(a) ab is positive.

(b) ab is negative.

(c) ab is positive and a is positive.

(d) ab is positive and a is negative.

(e) ab is negative and a is positive.

(f) ab is negative and a is negative.

9. Give the reason for each numbered step in the proof of the following theorem.

Theorem. If a and b are numbers such that both a and ab are positive, then b is also positive.

Proof. We assume that $0 < a$ and $0 < ab$. Then:

1. Exactly one of the following is true
   $b < 0$, $b = 0$, $0 < b$.
2. If $b = 0$, then $ab = 0$.
   Therefore, "$b = 0$" is false, since $ab = 0$ and $0 < ab$ are contradictory.
3. If $b < 0$, then $ab = -(|a||b|)$.
4. If $b < 0$, then ab is negative.
   Therefore, "$b < 0$" is false, since ab is negative and $0 < ab$ are contradictory.
   Therefore, $0 < b$; that is, b is positive, since this is the only remaining possibility.

Properties of Multiplication. The definition of multiplication for real numbers given in the preceding section was *suggested by* the structure properties which we wish to preserve for all numbers. On the other hand, we have not actually *assumed* these properties, since the definition could have been given at the outset without any reference to the properties. However, now that we have stated a definition for multiplication, it becomes important to satisfy ourselves that this definition really leads to the desired properties. In other words, we need to *prove* that multiplication so defined does have the properties. Since the definition is stated in terms of operations on positive numbers and 0 and of taking opposites, these operations are the only ones available to us in the proofs.

Multiplication property of 1: For any real number a,

$$a \cdot 1 = a.$$

Proof. If a is positive or 0, we know that $a(1) = a$. If a is negative, our definition of multiplication states that

$$\begin{aligned} a\cdot 1 &= -(|a|\cdot 1) \\ &= -|a| \\ &= a. \end{aligned}$$

Try to explain the reason why each step in the above proof is true.

Multiplication property of 0: For any real number a,

$$a\cdot 0 = 0.$$

Write out the proof of this property for yourself.

Commutative property of multiplication: For any real numbers a and b,

$$ab = ba.$$

Proof: If one or both the numbers a, b are zero, then $ab = ba$. (Why?) If a and b are both positive or both negative, then

$$ab = |a||b|, \text{ and } ba = |b||a|.$$

Since $|a|$ and $|b|$ are numbers of arithmetic,

$$|a||b| = |b||a|.$$

Hence,

$$ab = ba$$

for these two cases.

If one of a and b is positive or 0 and the other is negative, then

$$ab = -(|a||b|) \text{ and } ba = -(|b||a|).$$

Since

$$|a||b| = |b||a|,$$

and since if numbers are equal their opposites are equal,

$$-(|a||b|) = -(|b||a|).$$

Hence,

$$ab = ba$$

for this case also.

Here we have given a complete proof of the commutative property for all real numbers. We have based this proof on the precise definition of multiplication of real numbers.

Problems

1. Illustrate the proof of the commutative property by replacing a and b as follows:

(a) $(-3)(5)$ Example: $(-3)(5) = -(|-3||5|)$

$= -15$

$(5)(-3) = -(|5||-3|)$

$= -15$

(b) $(3)(-5)$ (d) $(-3)(-4)$

(c) $(-\frac{2}{3})(0)$ (e) $(-7)(\frac{5}{7})$

Associative property of multiplication: For any real numbers a, b, and c,

$$(ab)c = a(bc).$$

Proof: The property must be shown to be true for one negative, two negatives, or three negatives. This is lengthy, but we shall be able to simplify it by observing that

$$\begin{aligned} |(ab)c| &= |ab||c| \qquad \text{(Why?)} \\ &= |a||b||c|, \end{aligned}$$

and

$$\begin{aligned} |a(bc)| &= |a||bc| \\ &= |a||b||c|. \end{aligned}$$

Thus, $|(ab)c| = |a(bc)|$ for all real numbers a, b, c.

This reduces the proof of the associative property of multiplication to the problem of showing that $(ab)c$ and $a(bc)$ are either both positive, both zero, or both negative.

For example, if both $(ab)c$ and $a(bc)$ are negative, then $|(ab)c| = -\left(a(bc)\right)$ and $|a(bc)| = -\left((ab)c\right)$. Thus, $-\left(a(bc)\right) = -\left((ab)c\right)$ and hence, $a(bc) = (ab)c$.

If one of a, b, c is zero, then $(ab)c = 0$ and $a(bc) = 0$. (Why?) Hence, for this case $(ab)c = a(bc)$.

If a, b, and c are all different from zero, we need to consider eight different cases, depending on which numbers are positive and which are negative, as shown in the table below.

If a is	+	+	+	+	-	-	-	-
and b is	+	+	-	-	+	+	-	-
and c is	+	-	+	-	+	-	+	-
then ab is				-				
bc is				+				
$(ab)c$ is				+				
$a(bc)$ is				+				

One column has been filled in for positive a, and negative b and c. In this case, ab is negative and bc is positive. Hence, $(ab)c$ is positive and $a(bc)$ is positive. Therefore, $(ab)c = a(bc)$ in this case.

The associative property states that, in multiplying three numbers, we may first form the product of any adjacent pair. The effect of associativity along with commutativity is to allow us to write products of numbers without grouping symbols and to perform the multiplication in any groups and any orders.

Problems

1. Copy the table given in the text and complete it. Use it to check the remaining cases and finish the proof.
2. Explain how the associative and commutative properties can be used to perform the following multiplications in the easiest manner.

   (a) $(-5)(17)(-20)(3)$

   (b) $(-\frac{2}{3})(\frac{4}{5})(\frac{3}{2})(-\frac{5}{4})$

   (c) $(\frac{1}{5})(-19)(-3)(50)$

   (d) $(-7)(-25)(3)(-4)$

Another property is one which ties together the operations of addition and multiplication.

Distributive property. For any real numbers, a, b, and c,

$$a(b + c) = ab + ac.$$

We shall consider only a few examples:

$(5)\big(2 + (-3)\big) = ?$ and $(5)(2) + (5)(-3) = ?$
$(5)\big((-2) + (-3)\big) = ?$ and $(5)(-2) + (5)(-3) = ?$
$(-5)\big((-2) + (-3)\big) = ?$ and $(-5)(-2) + (-5)(-3) = ?$

The distributive property does hold for all real numbers. It could be proved by applying the definitions of multiplication and addition to all possible cases, but this is even more tedious than the proof of associativity.

Problems

Use the distributive property, if necessary, to perform the indicated operations with the minimum amount of work:

1. $(-9)(-92) + (-9)(-8)$
2. $(-\frac{3}{2})\big((-4) + 6\big)$
3. $(-7)(-\frac{3}{4}) + (-7)(\frac{1}{3})$
4. $(-\frac{3}{4})\big((-93) + (-7)\big)$

We can use the distributive property to prove another useful property of multiplication.

Theorem. For every real number a,

$$(-1)a = -a.$$

To prove this theorem, we must show that $(-1)a$ is the opposite of a, that is, that

$$a + (-1)a = 0.$$

Proof.

$$\begin{aligned} a + (-1)a &= 1(a) + (-1)a && \text{(Why?)} \\ &= \big(1 + (-1)\big)a && \text{(Why?)} \\ &= 0 \cdot a && \text{(Why?)} \\ &= 0. \end{aligned}$$

Here we have shown that $(-1)a$ is an additive inverse of a. Since we also know that $-a$ is an additive inverse of a and that the additive inverse is unique, we have proved that

$$(-1)a = -a.$$

Problems

Use the previous theorem to prove the following:

1. For any real numbers a and b, $(-a)(b) = -(ab)$.
2. For any real numbers a and b, $(-a)(-b) = ab$.
3. Write the common names for the products:

(a) $(-5)(ab)$

(b) $(-2a)(-5c)$

(c) $(3x)(-7y)$

(d) $(-0.5d)(1.2c)$

Use of the Multiplication Properties. We have seen in the previous problems that we may now write

$$-5a \text{ for } (-5)a \;,$$
$$-xy \text{ for } (x)(-y),$$
$$6b \text{ for } (-6)(-b).$$

In fact,

$$-ab = -(ab) = (-a)(b) = (a)(-b).$$

Now that we can multiply real numbers and have at our disposal the properties of multiplication of real numbers, we have a strong basis for dealing with a variety of situations in algebra.

Problems

1. Use the distributive property to write the following as indicated sums:

   (a) $3(x + 5)$
   (b) $2(a + b + c)$
   (c) $\big((-p) + q\big)(-3)$
   (d) $(-1)\big(y + (-z) + 5\big)$
   (e) $(13 + x)y$
   (f) $(-g)\big(r + 1 + (-s) + (-t)\big)$

2. Use the distributive property to write the following phrases as indicated products:

   (a) $5a + 5b$
   (b) $(-9)b + (-9)c$
   (c) $12 + 18$
   (d) $3x + 3y + 3z$
   (e) $(a + b)x + (a + b)y$
   (f) $7(\frac{1}{8}) + 3(\frac{1}{8})$
   (g) $(-6)a^2 + (-6)b^2$
   (h) $ca + cb + c$

3. Apply the distributive and other properties to the following:
   Example: $3x + 2x = (3 + 2)x = 5x$

   (a) $12t + 7t$
   (b) $9a + (-15a)$
   (c) $12z + z$ (Hint: $z = 1 \cdot z$)
   (d) $(-3m) + (-8m)$
   (e) $\frac{1}{2}a + \frac{3}{2}a$
   (f) $(1.6)b + (2.4)b$
   (g) $3a + 7y$ (Careful!)
   (h) $4p + 3p + 9p$
   (i) $6a + (-4a) + 5b + 14b$

In a phrase which has the form of an indicated sum $A + B$, A and B are called *terms* of the phrase; in a phrase of the form $A + B + C$, A, B, and C are called terms, etc. The distributive property is very helpful in simplifying a phrase. Thus, we found that

$$5a + 8a = (5 + 8)a = 13a$$

is a possible and a desirable simplification. However, in

$$5x + 8y$$

no such simplification is possible. Why?

We may sometimes be able to apply the distributive property to some, but not all, terms of an expression. Thus,

$$6x + (-9)x^2 + 11x^2 + 5y = 6x + \big((-9) + 11\big)x^2 + 5y = 6x + 2x^2 + 5y.$$

We shall have frequent occasion to do this kind of simplification. For convenience we shall call it *collecting terms* or *combining terms*. We shall usually do the middle step mentally. Thus,

$$15w + (-9)w = 6w.$$

Problems

1. Collect terms in the following phrases:
    - (a) $\frac{7}{8}a + \frac{9}{8}a$
    - (b) $5p + 4p + 8p$
    - (c) $7x + (-10x) + 3x$
    - (d) $12a + 5c + (-2c) + 3c^2$
    - (e) $6a + 4b + c$
    - (f) $9p + 4q + (-3)p + 7q$
    - (g) $4x + (-2)x^2 + (-5x) + 5x^2 + 1$
2. Find the truth set of each of the following open sentences. (Where possible, collect terms in each phrase.)
    - (a) $(-3a) + (-7a) = 40$
    - (b) $x + 5x = 3 + 6x$
    - (c) $3y + 8y + 9 = -90$
    - (d) $x + 2x + 3x = 42$
    - (e) $2y = y + 1$
    - (f) $12 = 4y + 2y$

__Further Use of the Multiplication Properties__. We have seen how the distributive property allows us to collect terms of a phrase. The properties of multiplication are helpful also in certain techniques of algebra related to products involving phrases.

__Example 1__. "$(3x^2y)(7ax)$" can be more simply written as "$21ax^3y$."

Give the reasons for each of the following steps which show this is true.

$$\begin{aligned}(3x^2y)(7ax) &= 3\cdot x\cdot x\cdot y\cdot 7\cdot a\cdot x\\ &= 3\cdot 7\cdot a\cdot x\cdot x\cdot x\cdot y\\ &= (3\cdot 7)a(x\cdot x\cdot x)y\\ &= 21ax^3y.\end{aligned}$$

(Notice that we write $x\cdot x\cdot x$ as x^3.)

While in practice we do not write down all these steps, __we must continue to be aware of how this simplification depends on our basic properties of multiplication__, and we should be prepared to explain the intervening steps at any time.

__Problems__

Simplify the following expressions and write the steps which explain the simplification.

1. $(\frac{3}{4}abc)(\frac{1}{2}bcd)$
2. $(20b^2c^2)(10bd)$
3. $(\frac{1}{3}ab)(9a^2)$
4. $(-7b)(-4a)c$

We can combine the method of the preceding exercises with the distributive property to perform multiplications such as the following:

$$(-3a)\big(2a + 3b + (-5)c\big) = (-6a^2) + (-9ab) + 15ac.$$

Furthermore, since we have shown in the properties of multiplication section that

$$-a = (-1)a,$$

we may again with the help of the distributive property simplify expressions such as

$$\begin{aligned} -\left(x^2 + (-7x) + (-6)\right) &= (-1)\left(x^2 + (-7x) + (-6)\right) \\ &= (-x^2) + 7x + 6. \end{aligned}$$

Problems

Write in the form indicated in the previous examples:

1. $2\left(8 + (-3b) + 7b^2\right)$
2. $6x(3y + z)$
3. $(-3)b^2c^2(4b + 7c)$
4. $10b\left(2b^2 + 7b + (-4)\right)$
5. $-(p + q + r)$
6. $(-7)\left(3a + (-5b)\right)$
7. $6xy(2x + 3xy + 4y)$
8. $(-x)\left(x + (-1)\right)$

Sometimes the distributive property is used several times in one example.

Example 1.

$$\begin{aligned} (x + 3)(x + 2) &= (x + 3)x + (x + 3)2 \\ &= x^2 + 3x + 2x + 6 \\ &= x^2 + (3 + 2)x + 6 \\ &= x^2 + 5x + 6 \end{aligned}$$

Example 2.

$$\begin{aligned} \left(a + (-7)\right)(a + 3) &= \left(a + (-7)\right)a + \left(a + (-7)\right)3 \\ &= a^2 + (-7)a + 3a + (-21) \\ &= a^2 + \left((-7) + 3\right)a + (-21) \\ &= a^2 + (-4)a + (-21) \end{aligned}$$

Example 3.

$$\begin{aligned} (x + y + z)(b + 5) &= (x + y + z)b + (x + y + z)5 \\ &= bx + by + bz + 5x + 5y + 5z. \end{aligned}$$

Problems

1. Perform the following multiplications.

   (a) $(x + 8)(x + 2)$

   (b) $\big(y + (-3)\big)\big(y + (-5)\big)$

   (c) $\big(6a + (-5)\big)\big(a + (-2)\big)$

   (d) $(a + 2)(a + 2)$

   (e) $(x + 6)\big(x + (-6)\big)$

   (f) $(y + 3)\big(y + (-3)\big)$

2. Show that for real numbers a, b, c, d,

   $$(a + b)(c + d) = ac + (bc + ad) + bd.$$

   (Notice that ac is the product of the first terms, bd is the product of the second terms, and $(bc + ad)$ is the sum of the remaining products.)

3. Multiply the following:

   (a) $(3a + 2)(a + 1)$

   (b) $(x + 5)(4x + 3)$

   (c) $(1 + n)(8 + 5n)$

   (d) $\big(2pq + (-8)\big)(3pq + 7)$

   (e) $\big(8 + (-3y) + (-y^2)\big)\big(2 + (-y)\big)$

   (f) $\big(5y + (-2x)\big)\big(3y + (-x)\big)$

Multiplicative Inverse. We have found that every real number has an additive inverse. In other words, for every real number there is another real number such that the sum of the two numbers is 0. Since a given real number remains unchanged when 0 is added to it (Why?), the number 0 is called the *identity element for addition*.

Is there a corresponding notion of multiplicative inverse for real numbers? First, we must have an identity element for multiplication. Since a given real number remains unchanged when it is multiplied by 1 (Why?), the number 1 is called the *identity element for multiplication*. For a given real number is there another real number such that the product of the two numbers is 1?

Consider, for example, the number 6. Is there a real number whose product with 6 is 1? By experiment or from your knowledge of arithmetic, you will probably say that $\frac{1}{6}$ is such a number, because $6 \cdot \frac{1}{6} = 1$. Find a number whose product with -2 is 1. Do the same for $-\frac{1}{3}$ and for $\frac{3}{4}$. Before going any further, let us write down a precise definition of multiplicative inverse.

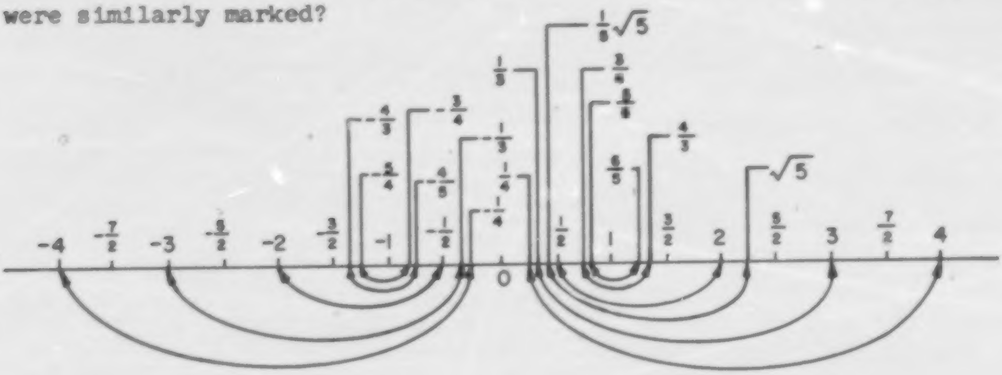
If c and d are real numbers, such that

$$cd = 1,$$

then d is called a multiplicative inverse of c.

If d is a multiplicative inverse of c, then is c a multiplicative inverse of d? Why? Does every real number have a multiplicative inverse? What is a multiplicative inverse of 0?

We can observe something of the way these inverses behave by looking at them on the number line. On the diagram below, some numbers and their inverses under multiplication are joined by double arrows. How can you test to see that these pairs of numbers really are multiplicative inverses? Can you visualize the pattern of the double arrows if a great many more pairs of these inverses were similarly marked?


$\frac{1}{5}\sqrt{5}$
$\frac{1}{3}$
$\frac{3}{4}$
$\frac{5}{6}$
$-\frac{4}{3}$
$-\frac{3}{4}$
$-\frac{1}{3}$
$\frac{4}{3}$
$-\frac{5}{4}$
$-\frac{4}{5}$
$\frac{1}{4}$
$\frac{6}{5}$
$\sqrt{5}$
$-\frac{1}{4}$
-4
$-\frac{7}{2}$
-3
$-\frac{5}{2}$
-2
$-\frac{3}{2}$
-1
$-\frac{1}{2}$
0
$\frac{1}{2}$
1
$\frac{3}{2}$
2
$\frac{5}{2}$
3
$\frac{7}{2}$
4

How about the number 0? With what number can 0 be paired? Is there a number b such that $0 \cdot b = 1$? What can you conclude about a multiplicative inverse of 0?

As you look at the double arrows in the above diagram, you may get the impression that the reciprocal of a number must be in a form obtained by interchanging numerator and denominator. What about $\sqrt{5}$? Certainly, $(\frac{1}{5}\sqrt{5}) \cdot \sqrt{5} = \frac{1}{5}(\sqrt{5} \cdot \sqrt{5}) = \frac{1}{5} \cdot 5 = 1$, so that $\sqrt{5}$ and $\frac{1}{5}\sqrt{5}$ are reciprocals.

The property toward which we have been working can now be stated. It really is a new property of the real numbers, since it cannot be derived as a consequence of the properties which we have stated up to this point.

> Existence of multiplicative inverses: For every real number c different from 0, there exists a real number d such that $cd = 1$.

That the real numbers actually have this property, if it is not already obvious to you, will become clearer as we do more problems. It is also obvious from experience that each non-zero number has exactly one multiplicative inverse; that is, the multiplicative inverse of a number is unique. We shall assume uniqueness, although it could be proved from the other properties, just as we did for the additive inverse.

Problems

1. Find the inverses under multiplication of the following numbers: $3, \frac{1}{2}, -3, -\frac{1}{2}, \frac{3}{4}, 7, \frac{5}{6}, -\frac{3}{7}, -7, \frac{3}{10}, \frac{1}{100}, -\frac{1}{100}, 0.45, -6.8$.
2. If b is a multiplicative inverse of a, what values for b do we obtain if a is larger than 1? What values of b do we obtain if a is between 0 and 1? What is a multiplicative inverse of 1?
3. If b is a multiplicative inverse of a, what values for b do we get if a is less than -1? If $a < 0$ and $a > -1$? What is a multiplicative inverse of -1?
4. For inverses under multiplication, what values of the inverse b do you obtain if a is positive? If a is negative?

Multiplication Property of Equality. We have previously stated the addition property of equality. Can we find a corresponding multiplication property? Consider the following statements:

Since $(-2)(3) = -6$, then $\big((-2)(3)\big)(-4) = (-6)(-4)$.
Since $(-5)(-3) = 15$, then $\big((-5)(-3)\big)(\frac{1}{3}) = (15)(\frac{1}{3})$.

Notice that "$(-2)(3)$" and "-6" are different names for the same number, and when we multiply (-4) by this number we obtain "$\big((-2)(3)\big)(-4)$" and "$(-6)(-4)$" as different names for a new number.

In general, we have the

Multiplication property of equality. For any real numbers a, b, and c, if $a = b$, then $ac = bc$.

Problems

1. Which of the following statements are true?

(a) If $2x = 6$, then $2x(\frac{1}{2}) = 6(\frac{1}{2})$.

(b) If $\frac{1}{3}a = 9$, then $\frac{1}{3}a(3) = 9(3)$.

(c) If $\frac{1}{4}n = 12$, then $\frac{1}{4}n(4) = 12(\frac{1}{4})$.

(d) If $\frac{2}{3}y = 16$, then $\frac{2}{3}y(\frac{3}{2}) = 16(\frac{3}{2})$.

(e) If $24 = \frac{3}{5}m$, then $24(\frac{3}{5}) = \frac{3}{5}m(\frac{3}{5})$.

2. Find the truth set of each of the following sentences.

Example: Determine the truth set of $\frac{5}{2}x = 60$.

If $\frac{5}{2}x = 60$ is true for some x,

then $\frac{5}{2}x(\frac{2}{5}) = 60(\frac{2}{5})$ is true for the same x;
(Why did we multiply by $\frac{2}{5}$?)

$\left((\frac{5}{2})(\frac{2}{5})\right)x = 24$ is true for the same x;

$x = 24.$

If $x = 24$,

then the left member is $\frac{5}{2}(24) = 60$, and the right member is 60.

So "$\frac{5}{2}(24) = 60$" is a true sentence, and the truth set is $\{24\}$.

(a) $12x = 6$

(b) $7x = 6$

(c) $\frac{2}{3}z = 1$

(d) $\frac{2}{3}z = \frac{2}{3}$

(e) $\frac{2}{3}z = \frac{3}{2}$

(f) $7a = 35$

(g) $\frac{1}{7}x = 5$

(h) $\frac{4}{9}c = -2$

(i) $5 = \frac{1}{3}a$

(j) $3 + x = -\frac{3}{2}$

(k) $0 = \frac{x}{12}$

Solutions of Equations. In the past, you found possible elements of truth sets of certain sentences, such as the equation

$$3x + 7 = x + 15,$$

and then checked these possibilities. Now we are prepared to solve such equations by a more general procedure. (To "solve" means to find the truth set.)

First, we know that any value of x for which

$$3x + 7 = x + 15$$

is true is also a value of x for which

$$(3x + 7) + \big((-x) + (-7)\big) = (x + 15) + \big((-x) + (-7)\big)$$

is true by the addition property of equality. The numerals in each member of this sentence can be simplified to give

$$\big(3x + (-x)\big) + \big(7 + (-7)\big) = \big(x + (-x)\big) + \big(15 + (-7)\big),$$
$$2x = 8.$$

Here we added the real number $\big((-x) + (-7)\big)$ to each member of the sentence and obtained the new sentence "$2x = 8$". Thus, each number of the truth set of "$3x + 7 = x + 15$" is a number of the truth set of "$2x = 8$", because the addition property of equality holds for all real numbers.

Next, we apply the multiplication property of equality to obtain

$$(\tfrac{1}{2})(2x) = (\tfrac{1}{2})(8),$$
$$x = 4.$$

Thus, each number of the truth set of "$2x = 8$" is a number of the truth set of "$x = 4$".

We can now deduce that every solution of "$3x + 7 = x + 15$" is a solution of "$x = 4$". The solution of the latter equation is obviously 4. But are we sure that 4 is a solution of "$3x + 7 = x + 15$"? We could easily check that it is, but let us use this example to suggest a general procedure.

The problem is this: We showed that, if x is a solution of

$$3x + 7 = x + 15,$$

then x is a solution of

$$x = 4.$$

What we must now show is that, if x is a solution of

$$x = 4,$$

then x is a solution of

$$3x + 7 = x + 15.$$

These two statements are usually written together as

> x is a solution of "$3x + 7 = x + 15$" _if_ _and_ _only_ _if_ x is a solution of "$x = 4$".

One way to show that the second of these statements is true is to reverse the steps in the proof of the first. Thus, if $x = 4$, we multiply by 2 to obtain

$$2x = 8.$$

(Notice that 2 is the reciprocal of $\frac{1}{2}$.) Then we add $(x + 7)$ to obtain

$$2x + (x + 7) = 8 + (x + 7),$$
$$3x + 7 = x + 15.$$

(Notice that $(x + 7)$ is the opposite of $\big((-x) + (-7)\big)$. Hence, every solution of "$x = 4$" is a solution of "$3x + 7 = x + 15$". That is, the one and only solution is 4.

> We say that "$x = 4$" and "$3x + 7 = x + 15$" are *equivalent sentences* in the sense that their truth sets are the same.

To summarize, if to both members of an equation we add a real number or multiply by a non-zero real number, the new sentence obtained is *equivalent* to the original sentence. This is true because these operations are "reversible". Then if we succeed in obtaining an equivalent sentence whose solution is obvious, we are sure that we have the required truth set *without checking*. Of course, a check may be desirable to catch mistakes in arithmetic.

As another example, solve the equation

$$5y + 8 = 2y + (-10).$$

This equation is equivalent to

$$(5y + 8) + \big((-2y) + (-8)\big) = \big(2y + (-10)\big) + \big((-2y) + (-8)\big),$$

that is, to

$$\big(5y + (-2y)\big) + \big(8 + (-8)\big) = \big(2y + (-2y)\big) + \big((-10) + (-8)\big)$$

and to

$$3y = -18.$$

In other words, y is a solution of "$5y + 8 = 2y + (-10)$" *if and only if* y is a solution of "$3y = -18$". The latter sentence is equivalent to

$$(\tfrac{1}{3})(3y) = (\tfrac{1}{3})(-18),$$

that is, to

$$y = -6.$$

Thus, y is a solution of "$3y = -18$" if and only if y is a solution of "$y = -6$". Hence, all three sentences are equivalent, and their truth set is $\{-6\}$. Here, we were certain that each step was reversible without actually doing it. When we solve an equation we ask ourselves at each step, "Is this step reversible?" If it is, we obtain an equivalent equation.

Later, we shall learn how to solve other types of sentences by means of applying properties of numbers. To do this we shall learn more about operations which yield equivalent sentences and others which do not.

Problems

1. Find the truth set of each of the following equations.

Example:

Solve: $(-3) + 4x = (-2x) + (-1)$.

This sentence is equivalent to

$$\big((-3) + 4x\big) + (2x + 3) = \big((-2x) + (-1)\big) + (2x + 3),$$

$$6x = 2;$$

and this is equivalent to

$$\frac{1}{6}(6x) = \frac{1}{6} \cdot 2,$$

$$x = \frac{1}{3}.$$

Hence, the truth set is $\{\frac{1}{3}\}$.

(a) $2a + 5 = 17$

(b) $12x + (-6) = 7x + 24$

(c) $8x + (-3x) + 2 = 7x + 8$ (Collect terms first.)

(d) $12n + 5n + (-4) = 3n + (-4) + 2n$

(e) $(-6a) + (-4) + 2a = 3 + (-a)$

(f) $0.5 + 1.5x + (-1.5) =$ [illegible]$5x + 2$

(g) $\frac{1}{2} + (-\frac{1}{2}c) + (-\frac{5}{2}) = 4c + (-2) + (-\frac{7}{2}c)$

(h) $(x + 1)(x + 2) = x(x + 2) = 3$

2. Translate the following into open sentences and find their truth sets, then answer the question in each problem.

   (a) The perimeter of a triangle is 44 inches. The second side is three inches more than twice the length of the third side, and the first side is five inches longer than the third side. Find the lengths of the three sides of this triangle.

   (b) If an integer and its successor are added, the result is one more than twice that integer. What is the integer?

   (c) The sum of two consecutive odd integers is 11. What are the integers?

   (d) Four times an integer is ten more than twice the successor of that integer. What is the integer?

   (e) In an automobile race, one driver, starting with the first group of cars, drove for 5 hours at a certain speed and was then 120 miles from the finish line. Another driver, who set out with a later heat, had traveled at the same rate as the first driver for 3 hours and was 250 miles from the finish. How fast were these men driving?

   (f) Plant A grows two inches each week, and it is now 20 inches tall. Plant B grows three inches each week, and it is now 12 inches tall. How many weeks from now will they be equally tall?

   (g) A number is increased by 17 and the sum is multiplied by 3. If the resulting product is 192, what is the number?

Reciprocals. We shall find it convenient to use the shorter name "reciprocal" for the multiplicative inverse, and we represent the reciprocal of a by the symbol "$\frac{1}{a}$". Thus, for every a except 0, $a \cdot \frac{1}{a} = 1$.

You probably noticed that for positive integers the symbol we chose for "reciprocal" is the familiar symbol of a fraction. Thus, the reciprocal of 5 is $\frac{1}{5}$. This certainly agrees with your former experience.

The reciprocal of $\frac{2}{3}$, however, is $\frac{1}{\frac{2}{3}}$; of -9 is $\frac{1}{-9}$; of 2.5 is $\frac{1}{2.5}$. Since $\frac{1}{\frac{2}{3}}$ is the reciprocal of $\frac{2}{3}$, and $\frac{2}{3} \times \frac{3}{2} = 1$, it follows that $\frac{1}{\frac{2}{3}}$ and $\frac{3}{2}$ must name the same number; since $\frac{1}{-9}$ is the reciprocal of -9 and since $-9 \times (-\frac{1}{9}) = 1$, $\frac{1}{-9}$ and $-\frac{1}{9}$ must name the same number. Since $(2.5)(0.4) = 1$, 0.4 and $\frac{1}{2.5}$ must name the same number. We shall be in a better position to continue this discussion after we consider division of real numbers in a later section.

Problems

1. If the reciprocal of any non-zero number a is represented by the symbol "$\frac{1}{a}$", represent the reciprocal of:

(a) 15	(e) $\frac{5}{3}$
(b) -8	(f) 0.3
(c) $\frac{1}{5}$	(g) $-\frac{3}{4}$
(d) $-\frac{1}{6}$	

2. Prove the theorem: For each non-zero real number a there is only one multiplicative inverse of a. (Hint: We know there *is* a multiplicative inverse of a, namely, $\frac{1}{a}$. Assume there is *another*, say x. Then $ax = 1$.)

Why did we exclude 0 from our definition of reciprocals? Suppose 0 did have a reciprocal. What could it be? If there *were* a number b which is the reciprocal of 0, then $0 \cdot b = 1$. What is the truth set of the sentence $0 \cdot b = 1$? You should conclude that 0 simply cannot have a reciprocal. Here we have an opportunity to demonstrate, using a rather simple example, a very powerful type of proof. This proof depends on the fact that a given sentence *is either true or it is false*, *but not both*. An assertion that a given sentence is *both* true and false is called a *contradiction*. If we reach a contradiction in a chain of correct reasoning, then we are forced to admit that the reasoning is based on a false statement. This is the idea behind the proof of the next theorem.

Theorem. The number 0 has no reciprocal.

Proof. The sentence "0 has no reciprocal" is either true or false, but not both. Assuming that it is *false*, that is, assuming that 0 *does* have a reciprocal, we have the following chain of reasoning:

1.	There is a real number a such that $0 \cdot a = 1$.	Assumption.
2.	$0 \cdot a = 0$	The product of 0 with any real number is 0.
3.	Therefore $0 = 1$.	

Thus, we are led to the assertion that "$0 = 1$" is a true sentence. But "$0 = 1$" is obviously a false sentence, and so we have a contradiction. Conclusion: Step 1 of the argument cannot be true. Therefore, it is false that 0 has a reciprocal; that is, 0 has *no* reciprocal.

A proof of the above type is called *indirect* or a *proof by contradiction* or a *reductio ad absurdum*.

We should like now to see what we can discover and what we can prove about the way reciprocals behave.

In each of the following sets of numbers, find the reciprocals. What conclusion do you draw about reciprocals on examining the two sets?

$$\text{I:}\quad 12,\ \frac{1}{8},\ 150,\ 0.09,\ \frac{8}{9}$$

$$\text{II:}\quad -5,\ -\frac{1}{3},\ -700,\ -2.2,\ -\frac{5}{3}$$

Observation of reciprocals on the number line strengthens our belief that the following theorem is true.

Theorem. The reciprocal of a positive number is positive, and the reciprocal of a negative number is negative.

Proof. We know that $a \cdot \frac{1}{a} = 1$; that is, the product of a non-zero number and its reciprocal is the positive number 1. Let us first assume that a is positive. Then exactly one of three possibilities must be true:

$$\frac{1}{a} < 0,\quad \frac{1}{a} = 0,\quad 0 < \frac{1}{a}.$$

We see that if $\frac{1}{a}$ is negative, then $a \cdot \frac{1}{a}$ is negative, a contradiction of the fact that $a \cdot \frac{1}{a}$ is positive. Also, if $\frac{1}{a} = 0$ when a is positive, then $a \cdot \frac{1}{a} = 0$, again a contradiction. This leaves but one remaining possibility: $\frac{1}{a}$ is positive.

In the same way, we may show that if a is negative, then $\frac{1}{a}$ is negative.

For each of the following numbers, find the reciprocal of the number; then find the reciprocal of that reciprocal. What conclusion is suggested?

$$-12,\ 80,\ \frac{19}{20},\ -\frac{1}{9},\ 1.6$$

Theorem. The reciprocal of the reciprocal of a non-zero real number a is a itself.

Proof. The reciprocal of the reciprocal of a is $\frac{1}{\frac{1}{a}}$. Since $\frac{1}{\frac{1}{a}}$ and a are both reciprocals of $\frac{1}{a}$ (why?), and since there is exactly one reciprocal of $\frac{1}{a}$, it follows that "$\frac{1}{\frac{1}{a}}$" and "a" must be names of the same number. Hence,

$$\frac{1}{\frac{1}{a}} = a.$$

Problems

1. For what real values of a do the following numbers have no reciprocals?

   $a + (-1)$, $a + 1$, $a^2 + (-1)$, $a(a + 1)$, $\frac{a}{a + 1}$, $a^2 + 1$, $\frac{1}{a^2 + 1}$

2. Consider the sentence

   $$\big(a + (-3)\big)(a + 1) = a + (-3),$$

   which has the truth set $\{0, 3\}$. (Verify this fact.) If both members of the sentence are multiplied by the reciprocal of $a + (-3)$, that is, by $\frac{1}{a + (-3)}$, and some properties of real numbers are used (which properties?), we obtain the sentence

   $$a + 1 = 1.$$

   For $a = 3$, we have $3 + 1 = 1$, and this is clearly a false sentence. Why doesn't the new sentence have the same truth set as the original sentence?

3. Consider three pairs of numbers: (1) $a = 2$, $b = 3$; (2) $a = 4$, $b = -5$; (3) $a = -4$, $b = -7$. Does the sentence $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$ hold true in all three cases?

4. Is the sentence $\frac{1}{a} > \frac{1}{b}$ true in all three cases of Problem 3?

5. Is it true that if $a > b$ and a, b are positive, then $\frac{1}{b} > \frac{1}{a}$? Try this for some particular values of a and b.

6. Is it true that if $a > b$ and a, b are negative, then $\frac{1}{b} > \frac{1}{a}$? Substitute some particular values of a and b.

7. Could you tell immediately which reciprocal is greater than another if one of the numbers is positive and the other is negative?

In Problem 3 on the previous page you showed that for three particular pairs of values of a and b, $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$. In other words, the product of the reciprocals of these two numbers is the reciprocal of their product. How many times would we need to test this sentence for particular numbers in order to be sure it is true for all real numbers except zero? Would 1,000,000 tests be enough? How would we know that the sentence would not be false for the 1,000,001st test?

We can often reach probable conclusions by observing what happens in a number of particular cases. We call this inductive reasoning. No matter how many cases we observe, inductive reasoning alone cannot assure us that a statement is always true.

Thus, we cannot use inductive reasoning to prove that $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$ is always true. We can prove it for all non-zero real numbers by deductive reasoning as follows. (Remember that in the proof we may use only properties which have already been stated.)

Theorem. For any non-zero real numbers a and b,

$$\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}.$$

Discussion: Since our object is to prove that $\frac{1}{a} \cdot \frac{1}{b}$ is the reciprocal of ab, we recall the definition of reciprocal. Then we concentrate on the product $ab \cdot (\frac{1}{a} \cdot \frac{1}{b})$ and try to show that it is 1.

Proof. $ab \cdot (\frac{1}{a} \cdot \frac{1}{b}) = a(\frac{1}{a}) \cdot b(\frac{1}{b})$ by commutative and associative properties

$= 1 \cdot 1$ since $x \cdot \frac{1}{x} = 1$

$= 1.$

Hence, $\frac{1}{a} \cdot \frac{1}{b}$ is the reciprocal of ab. In other words,

$$\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}.$$

Notice how closely the proof of this theorem parallels the proof that the sum of the opposites of two numbers is the opposite of their sum. Remember how this result was proved:
$(a + b) + \Big((-a) + (-b)\Big) = \Big(a + (-a)\Big) + \Big(b + (-b)\Big) = 0$; hence, $(-a) + (-b) = -(a + b)$.

Problems

1. Do the following multiplications. (In these and in future problem sets we assume that the values of the variables are such that the fractions have meaning.)

   (a) $(\frac{1}{2a})(\frac{1}{3b})$

   (b) $(\frac{1}{-3y})(\frac{1}{-7z})$

   (c) $(\frac{1}{\frac{1}{3}ab})(\frac{1}{9a^2})$

   (d) $(\frac{1}{-2m^2n})(-\frac{1}{3mn^2})$

   (e) $(\frac{1}{x})(\frac{1}{\frac{1}{x}})$

2. Is $8 \cdot 17 = 0$ a true sentence? Why?

3. If $n \cdot 50 = 0$, what can you say about n?

4. If $p \cdot 0 = 0$, what can you say about p?

5. If $p \cdot q = 0$, what can you say about p or q?
6. If $p \cdot q = 0$, and we know that $p > 10$, what can we say about q?

The idea suggested by the above exercises will be a very useful one, especially in finding truth sets of certain equations. We are able to prove the following theorem now by using the properties of reciprocals.

Theorem. For real numbers a and b, $ab = 0$ if and only if $a = 0$ or $b = 0$.

Because of the "if and only if", we really must prove two theorems: (1) If $a = 0$ or $b = 0$, then $ab = 0$; (2) If $ab = 0$, then $a = 0$ or $b = 0$.

Proof. If $a = 0$ or $b = 0$, then $ab = 0$ by the multiplication property of 0. Thus, we have proved one part of the theorem.

To prove the second part of the theorem, note that either $a = 0$ or $a \neq 0$, but not both. If $a = 0$, the requirement that $a = 0$ or $b = 0$ is satisfied. Why?
If $ab = 0$ and $a \neq 0$, then there is a reciprocal of a and

$$\left(\frac{1}{a}\right)(ab) = \frac{1}{a} \cdot 0, \qquad \text{(Why?)}$$

$$\left(\frac{1}{a}\right)(ab) = 0, \qquad \text{(Why?)}$$

$$\left(\frac{1}{a} \cdot a\right)b = 0, \qquad \text{(Why?)}$$

$$1 \cdot b = 0, \qquad \text{(Why?)}$$

$$b = 0.$$

Thus, in this case also the requirement that $a = 0$ or $b = 0$ is satisfied; hence, we have proved the second part of the theorem.

Problems

1. If $\big(x + (-5)\big) \cdot 7 = 0$, what must be true about 7 or $\big(x + (-5)\big)$? Can 7 be equal to 0? What about $x + (-5)$ then?

2. Explain how we know that the only value of y which will make $9 \times y \times 17 \times 3 = 0$ a true sentence is 0.

3. If a is between p and q, is $\frac{1}{a}$ between $\frac{1}{p}$ and $\frac{1}{q}$? Explain.

4. This last theorem enables us to determine the truth set of an equation such as

$$\big(x + (-3)\big)\big(x + (-8)\big) = 0$$

without guesswork. With $a = \big(x + (-3)\big)$ and $b = \big(x + (-8)\big)$, the theorem tells us that this sentence is equivalent to the sentence

$$x + (-3) = 0 \text{ or } x + (-8) = 0.$$

From this sentence we read off the truth set as $\{3, 8\}$. Find the truth set of each of the following equations:

(a) $\big(x + (-20)\big)\big(x + (-100)\big) = 0$

(b) $(x + 6)(x + 9) = 0$

(c) $x\big(x + (-4)\big) = 0$

(d) $\big(3x + (-5)\big)\ \big(2x + (-1)\big) = 0$

(e) $\big(x + (-1)\big)\big(x + (-2)\big)\big(x + (-3)\big) = 0$

(f) $2\big(x + (-\frac{1}{2})\big)\ (x + \frac{3}{4}) = 0$

(g) $9|x + (-6)| = 0$

(h) $x(x + 4) = x^2 + 8$

5. Prove: If a, b, c are real numbers, and if $ac = bc$ and $c \neq 0$, then $a = b$.

The Two Basic Operations and the Inverse of a Number Under These Operations. We have focused our attention on addition and multiplication and on the inverses under these two operations. These four concepts are basic to the real number system. Addition and multiplication have a number of properties by themselves, and one property connects addition with multiplication, namely, the distributive property. All our work in algebraic simplification rests on these properties and on the various consequences of them which relate addition, multiplication, opposite, and reciprocal.

We have pointed out that the distributive property connects addition and multiplication. It is instructive to see whether some relationship occurs which connects every combination of addition, multiplication, opposite, and reciprocal in pairs. Let us write down all possible combinations.

1. Addition and multiplication: The distributive property, $a(b + c) = ab + ac$; for example,
   $2(3 + 5) = 2 \cdot 3 + 2 \cdot 5 = 6 + 10 = 16$;
   $4\big(7 + (-2)\big) = 4 \cdot 7 + 4(-2) = 28 - 8 = 20$.
2. Addition and opposite: We have proved that
   $-(a + b) = (-a) + (-b)$; for example,
   $-(3 + 5) = (-3) + (-5) = (-8)$;
   $-\big(7 + (-2)\big) = (-7) - (-2) = -5$.
3. Addition and reciprocal: We find that there *is no simple* relationship connecting $\frac{1}{a} + \frac{1}{b}$ and $\frac{1}{a + b}$; for example, $\frac{1}{3} + \frac{1}{2}$ does *not* equal $\frac{1}{5}$. In fact, there are no real numbers at all for which these two phrases represent the same number. This unfortunate lack of relationship is considerable cause of trouble in algebra for students who unthinkingly assume that these expressions represent the same number.
4. Multiplication and opposite: We have proved that
   $-(ab) = (-a)(b) = (a)(-b)$; for example,
   $-(2 \cdot 5) = (-2)(5) = (2)(-5) = -10$;
   $-\big((-3)(4)\big) = (3)(4) = (-3)(-4) = 12$.

5. Multiplication and reciprocal: We have proved that $\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}$; for example, $\frac{1}{5 \cdot 6} = \frac{1}{5} \cdot \frac{1}{6}$.
6. Opposite and reciprocal: $\frac{1}{(-a)} = -(\frac{1}{a})$.

   This last relation is a new one and should be proved. The proof may be obtained from (5) above by replacing b by -1. The proof is left to the students. (Hint: What is the reciprocal of -1?)

State (1), (2), (4), (5), (6), in words. Do you see any similarity between addition and opposite, on the one hand, and multiplication and reciprocal, on the other, in these properties? Explain.

At this point, we should be sure that we are familiar with the more important ideas of this section. Can you summarize these ideas about real numbers?

Part 4

PROPERTIES OF ORDER

The Order Relation for Real Numbers. In Part III we extended the concept of order from the numbers of arithmetic to all real numbers. This was done by using the number line, and we agreed that:

> "Is less than", for real numbers, means "to the left of" on the real number line.
>
> If a and b are real numbers, then "a is less than b" is written "$a < b$".

We speak of the relation "is less than" for real numbers as an order relation. It is a binary relation since it expresses a relation between two numbers.

Two basic properties of the order relation for real numbers were obtained in Part III.

> Comparison property: If a is a real number, then exactly one of the following is true:
>
> $$a < b, \quad a = b, \quad b < a.$$
>
> Transitive property: If a, b, c are real numbers and if $a < b$ and $b < c$, then $a < c$.

Another property of order which was obtained in Part III connects the order relation with the operation of taking opposites:

> If a and b are real numbers and if $a < b$, then $-b < -a$.

You may wonder at this point why we are so careful to avoid talking about "greater than". As a matter of fact, the relation "is greater than", for which we use the symbol "$>$", is also an order relation. Does this order relation have the comparison property and the transitive property? Since it does, we actually have two different (though very closely connected!) order relations for the real numbers, and we have chosen to concentrate our attention on "less than". We could have decided to concentrate on "greater than"; but if we are going to study an order relation and its properties, we must not confuse the issue by shifting from one order relation to another in the middle of the discussion. We need not concern ourselves unduly with this problem since any sentence involving ">" can be written as one involving "<". Does "$a < b$" imply that "$b > a$"? Try this with some unequal pairs of real numbers.

Thus, we state the last property mentioned above in terms of "<", but in applying the property we feel free to say, "If $a < b$, then $-a > -b$".

In the next two sections we obtain some properties of the order relation "<" which involve the operations of addition and multiplication. Such properties are essential if we are to make much use of the order relation in algebra.

Problems

1. For each pair of numbers, determine their order.

   (a) $-\frac{3}{2}$, $-\frac{4}{3}$ (b) $-|-7|$, $-|7|$

   (c) c, 1 (Consider the comparison property.)

2. Continuing Problem 1(c), what can you say about the order of c and 1 if it is known that $c > 4$? What property of order did you use here?

3. Decide in each case whether the sentence is true.

   (a) $-3 + (-2) < 2 + (-2)$

   (b) $(-3) + (0) < 2 + 0$

   (c) $(-3) + 5 < 2 + 5$

   (d) $(-3) + a < 2 + a$

   (e) $(-3)(5) < (2)(5)$

   (f) $(-3)(0) < (2)(0)$

   (g) $(-3)(-2) < (2)(-2)$

   (h) $(-3)(a) < (2)(a)$ (What is the truth set of this sentence?)

(i) $|4|(3 + (-2)) < |-6|(2 + (-3))$

4. A given set may be described in many ways. Describe in three ways the truth set of:

(a) $3 < 3 + x$ (b) $3 + x < 3$

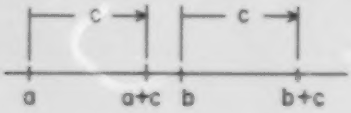
5. Determine the truth set of

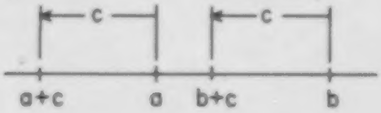
(a) $y < 3$ (c) $-|y| < 3$

(b) $|y| < 3$ (d) $-y < 3$

Addition Property of Order. What is the connection between order of numbers and addition of numbers? We shall find a basic property, and from it prove other properties which relate order and addition. As before, we concentrate on the order relation "<"; similar properties can be stated for the order relation ">".

It is helpful to view addition and order on the number line. We remember that adding a positive number means moving to the right; adding a negative number means moving to the left. Let us fix *two* points a and b on the number line, with $a < b$, and, hence, a to the *left* of b. If we add the same number c to a and to b, we move to the right of a and of b if c is *positive*, to the left if c is *negative*. Whether c is positive or negative we see from the figure that $a + c$ is still to the left of $b + c$.





Here we have found a fundamental property of order which we shall assume for all real numbers.

> *Addition Property of Order*. If a, b, c are real numbers and if $a < b$, then
> $$a + c < b + c.$$

Illustrate this property for $a = -3$ and $b = -\frac{1}{2}$, with c having, successively, the values -3, $\frac{1}{2}$, 0, -7. Here, $-3 < -\frac{1}{2}$. Is "$(-3) + (-3) < (-\frac{1}{2}) + (-3)$" a true sentence? Continue with the other values of c. Phrase the addition property of order in words. Is there a corresponding property of equality?

Problems

1. By applying the addition properties of order, determine which of the following sentences are true.

   (a) $(-\frac{6}{5}) + 4 < (-\frac{3}{4}) + 4$

   (b) $(-\frac{5}{3})(\frac{6}{5}) + (-5) > (-\frac{5}{2}) + (-5)$

   (c) $(-5.3) + (-2)(-\frac{4}{3}) < (-0.4) + \frac{8}{3}$

   (d) $(\frac{5}{2})(-\frac{3}{4}) + 2 \geq (-\frac{15}{8}) + 2$

2. Formulate an addition property of order for each of the relations "$\leq$", "$>$", "$\geq$".

3. An extension of the order property states that:

   If a, b, c, d are real numbers, such that $a < b$ and $c < d$, then $a + c < b + d$.

   This can be proved in three steps. Give the reason for each step:

   If $a < b$, then $a + c < b + c$;
   if $c < d$, then $b + c < b + d$;
   hence,

   $$a + c < b + d.$$

4. Find the truth set of each of the following sentences.

Example: If $(-\frac{3}{2}) + x < (-5) + \frac{3}{2}$ is true for some x,

then $x < (-5) + \frac{3}{2} + \frac{3}{2}$ is true for the same x.

$x < -2$ is true for the same x.

Thus, if x is a number which makes the original sentence true, then $x < -2$. If "$x < -2$" is true for some x, then

$(-\frac{3}{2}) + x < (-\frac{3}{2}) + (-2)$ is true for the same x,

$(-\frac{3}{2}) + x < (-\frac{3}{2}) + \left((-5) + 3\right)$,

$(-\frac{3}{2}) + x < (-5) + \left(3 + (-\frac{3}{2})\right)$,

$(-\frac{3}{2}) + x < (-5) + \frac{3}{2}$ is true for the same x.

Hence, the truth set is the set of all real numbers less than -2.

(a) $3 + x < (-4)$

(b) $x + (-2) > -3$

(c) $2x < (-5) + x$

(d) $3x > \frac{4}{3} + 2x$

(e) $(-\frac{2}{3}) + 2x \geq \frac{5}{3} + x$

(f) $(-x) + 4 < (-3) + |-3|$

(g) $(-5) + (-x) < \frac{2}{3} + |-\frac{4}{3}|$

(h) $(-2) + 2x < (-3) + 3x + 5$

(i) $(-\frac{3}{4}) + \frac{5}{4} \geq x + |-\frac{3}{2}|$

5. Graph the truth sets of Parts (a), (c), and (h) of Problem 4.

6. The following property of order has been previously stated: "If $a < b$, then $-b < -a$." Prove this property, using the addition property of order. (Hint: Add $\left((-a) + (-b)\right)$ to both members of the inequality $a < b$; then use property of additive inverses.)

7. Show that the property:

"If $0 < y$, then $x < x + y$."

is a special case of the addition property of order. (Hint: In the statement of the addition property of order, let $a = y$, $b = 0$, $c = x$.)

Many results about order can be proved as consequences of the addition property of order. Two of these are of special interest to us, because they give direct translations back and forth between statements about order and statements about equality.

The first of these results will be a special case of the property. Let us consider a few numerical examples of the property with $a = 0$. If $a = 0$, then "$a < b$" becomes "$0 < b$"; that is, b is a positive number. Thus, we may write: If $0 < b$, then $c + 0 < c + b$.

Let $a = 0$, $b = 3$ and $c = 4$; then $4 + 0 < 4 + 3$; that is, since $7 = 4 + 3$, then $4 < 7$.

Let $a = 0$, $b = 5$ and $c = -4$; then $(-4) + 0 < (-4) + 5$; that is, since $1 = (-4) + 5$, then $-4 < 1$.

These two examples can be thought of as saying:

Since $7 = 4 + 3$ and 3 is a positive number, then $4 < 7$.

Since $1 = (-4) + 5$ and 5 is a positive number, then $-4 < 1$.

This result we state as

Theorem. If $z = x + y$ and y is a positive number, then $x < z$.

Proof. We may change the addition property of order to read:

If $a < b$, then $c + a < c + b$. (Why?)

Since the property is true for all real numbers a, b, c, we may let $a = 0$, $b = y$, $c = x$. Thus,

if $0 < y$, then $x + 0 < x + y$.

If $z = x + y$, then "$x + 0 < x + y$" means "$x < z$". Hence, we have proved that if $z = x + y$ and $0 < y$ (y is positive) then $x < z$.

This theorem now gives us a translation from a statement about equality, such as

$$-4 = (-6) + 2,$$

to a statement about order, in this case

$$-6 < -4.$$

Notice that adding 2, a positive number, to (-6) yields a number to the right of -6.

Change the sentence

$$4 = (-2) + 6$$

to a sentence involving order.

The second result of the addition property is a theorem which translates from order to equality, instead of from equality to order. You have seen that if y is positive and x is any number, then x is always less than $x + y$. If $x < z$, then does there exist a positive number y such that $z = x + y$? Consider, for example, the numbers 5 and 7 and note that $5 < 7$. What is the number y such that

$$7 = 5 + y?$$

How did you determine y? Did you find y to be positive? Consider the numbers -3 and -6, noting that $-6 < -3$. What is the truth set of

$$-3 = -6 + y?$$

Is y again positive?

$4 < 9$, $\quad 9 = 4 + (\ )$

$-3 < 5$, $\quad 5 = (-3) + (\ )$

$-4 < -1$, $\quad -1 = (-4) + (\ )$

$-6 < 0$, $\quad 0 = (-6) + (\ )$

What kind of number makes each of the above equations true? In each case you added a positive number to the smaller number to get the greater.

By this time you see that the theorem we have in mind is

> *Theorem*. If x and z are two real numbers such that $x < z$, then there is a positive real number y such that
> $$z = x + y.$$

Proof. There are really two things to be proved. First, we must *find* a value of y such that $z = x + y$; second, we must prove that the y we found is *positive*, if $x < z$.

It is not hard to find a value of y such that $z = x + y$. Your experience with solving equations probably suggests adding $(-x)$ to both members of "$z = x + y$" to obtain $y = z + (-x)$.

Let us try this value of y. Let

$$y = z + (-x).$$

Then

$$\begin{aligned} x + y &= x + \big(z + (-x)\big) && \text{(Why?)} \\ &= \big(x + (-x)\big) + z && \text{(Why?)} \\ &= 0 + z. \\ x + y &= z \end{aligned}$$

Thus, we have found a y, namely, $z + (-x)$, such that $z = x + y$.

It remains to show that if $x < z$, then this y is positive. We know there is exactly one true sentence among these: y is negative, y is zero, y is positive. (Why?) If we can show that two of these possibilities are false, the third *must* be true. Try the first possibility: If it were true that y is *negative* and $z = x + y$, then the addition property of order would assert that $z < x$. (Let $a = y$, $b = 0$, $c = x$.) But this contradicts the fact that $x < z$; so it cannot be true that y is negative. Try the second possibility: If it were true that y is *zero* and $z = x + y$, then it would be true that $z = x$. This again contradicts the fact that $z < x$; so it cannot be true that y is zero. Hence, we are left with only one possibility, y is *positive*, which must be true. This completes the proof.

This theorem allows us to translate from a sentence involving order to one involving equality. Thus,

$$-5 < -2$$

can be replaced by

$$-2 = (-5) + 3,$$

which gives the same "order information" about -5 and -2. That is, there is a positive number, 3, which when added to the lesser, -5, yields the greater, -2.

Problems

1. For each pair of numbers, determine their order and find the positive number b which when added to the smaller gives the larger.

   (a) -15 and -24

   (b) $\frac{63}{4}$ and $-\frac{5}{4}$

   (c) $\frac{6}{5}$ and $\frac{7}{10}$

   (d) $-\frac{1}{2}$ and $\frac{1}{3}$

   (e) -254 and -345

   (f) $-\frac{33}{13}$ and $-\frac{98}{39}$

   (g) 1.47 and -0.21

   (h) $(-\frac{2}{3})(\frac{4}{5})$ and $(\frac{3}{2})(-\frac{5}{4})$

2. Show that the following is a true statement: If a and c are real numbers and if $c < a$, then there is a negative real number b such that $c = a + b$. (Hint: Follow the similar discussion for b positive.)

3. Which of the following sentences are true for all real values of the variables?

   (a) If $a + 1 = b$, then $a < b$.

   (b) If $a + (-1) = b$, then $a < b$.

   (c) If $(a + c) + 2 = (b + c)$, then $a + c < b + c$.

   (d) If $(a + c) + (-2) = (b + c)$, then $b + c < a + c$.

(e) If $a < -2$, then there is a positive number d such that $-2 = a + d$.

(f) If $-2 < a$, then there is a positive number d such that $a = (-2) + d$.

4. (a) Use $5 + 8 = 13$ to suggest two true sentences involving "<" relating pairs of the numbers 5, 8, 13.

   (b) Since $(-3) + 2 = (-1)$, how many true sentences involving "<" can you write using pairs of these three numbers?

   (c) If $5 < 7$, write two true sentences involving "=" relating the numbers 5, 7.

5. Show on the number line that if a and c are real numbers and if b is a negative number such that $c = a + b$, then $c < a$.

6. Which of the following sentences are true for all values of the variables?

   (a) If $b < 0$, then $3 + b < b$.

   (b) If $b < 0$, then $3 + b < 3$.

   (c) If $x < 2$, then $2x < 4$.

7. Verify that each of the following is true.

   (a) $|3 + 4| \leq |3| + |4|$

   (b) $|(-3) + 4| \leq |-3| + |4|$

   (c) $|(-3) + (-4)| \leq |-3| + |-4|$

   (d) State a general property relating $|a + b|$, $|a|$ and $|b|$ for any real numbers a and b.

8. What general property can be stated for multiplication similar to the property for addition in Problem 7?

9. Translate the following into open sentences and find their truth sets.

   (a) The sum of a number and 5 is less than twice the number. What is the number?

   (b) When Joe and Moe were planning to buy a sailboat, they asked a salesman about the cost of a new type of a boat that was being designed. The salesman replied, "It won't cost more than \$380." If Joe and Moe had agreed that Joe was to contribute \$130 more than Moe when the boat was purchased, how much would Moe have to pay?

   (c) Three more than six times a number is greater than seven increased by five times the number. What is the number?

   (d) A teacher says, "If I had twice as many students in my class as I do have, I would have at least 26 more than I now have." How many students does he have in his class?

   (e) A student has test grades of 82 and 91. What must he score on a third test to have an average of 90 or higher?

   (f) Bill is 5 years older than Norman, and the sum of their ages is less than 23. How old is Norman?

Multiplication Property of Order. We have stated a basic property giving the order of $a + c$ and $b + c$ when $a < b$. Let us now ask about the order of the products ac and bc when $a < b$.

Consider the true sentence

$$5 < 8.$$

If each of these numbers is multiplied by 2, the products are involved in the true sentence

$$(5)(2) < (8)(2).$$

What is your conclusion about a multiplication property of order? Before making a decision, let us try more examples. Just as above, where we took the two numbers 5 and 8 in the true sentence

"$5 < 8$" and inserted them in "$(\ )(2) < (\ )(2)$" to make a true sentence, do the same in the following.

1. $7 < 10$ and $(\ )(6) < (\ )(6)$
2. $-9 < 6$ and $(\ )(5) < (\ )(5)$
3. $2 < 3$ and $(\ )(-4) < (\ )(-4)$
4. $-7 < -2$ and $(\ )(2) < (\ )(2)$
5. $-1 < 8$ and $(\ )(-3) < (\ )(-3)$
6. $-5 < -4$ and $(\ )(-6) < (\ )(-6)$

We are concerned here with the order relation "$<$", observing the pattern when each of the numbers in the statement "$a < b$" is multiplied by the same number. Did you notice that it makes a difference whether we multiply by a positive number or a negative number?

The above experience suggests that if $a < b$, then

$ac < bc$, *provided* c *is a positive number*;
$bc < ac$, *provided* c *is a negative number*.

Thus, we have found another important set of properties of order.

How can you use these properties to tell quickly whether the following sentences are true?

Since $\frac{1}{4} < \frac{2}{7}$, then $\frac{5}{4} < \frac{10}{7}$.

Since $-\frac{5}{6} < -\frac{14}{17}$, then $\frac{14}{51} < \frac{5}{18}$.

Since $\frac{5}{3} < \frac{7}{4}$, then $-\frac{7}{16} < -\frac{5}{12}$.

These properties of order turn out to be consequences of the other properties of order, and we state them together as

Theorem. *Multiplication Property of Order*.
If a, b, and c are real numbers and if $a < b$, then

$ac < bc$, if c is positive,
$bc < ac$, if c is negative.

Proof. There are two cases. Let us consider the case of *positive* c. Here we must prove that if $a < b$, then $ac < bc$.

Fill in the reason for each step of the proof.

1. There is a positive number d such that $b = a + d$.
2. Therefore, $bc = (a + d)c$.
3. $bc = ac + dc$.
4. The number dc is positive.
5. Hence, $ac < bc$.

The proof of the case for *negative* c is left to the student in the problems.

We could equally well have discussed the multiplication property of the order relation "is greater than" instead of "is less than".

When we are comparing numbers, the two statements "$a < b$" and "$b > a$" say the same thing about a and b. Thus, when we are concerned primarily with numbers rather than a particular order relation, it may be convenient to shift from one order relation to another and write such sentences as:

Since $3 < 5$, then $3(-2) > 5(-2)$.
Since $-2 > -5$, then $(-2)(8) > (-5)(8)$.
Since $3 > 2$, then $(3)(-7) < (2)(-7)$.

Verify that these sentences are true.

When we are focusing on the numbers involved instead of on an order relation, we can say that

$$\text{if } a < b, \text{ then } \begin{cases} ac < bc & \text{if } c \text{ is positive,} \\ ac > bc & \text{if } c \text{ is negative.} \end{cases}$$

State these properties of orders in your own words.

In our study we shall also need some results such as

Theorem. If $x \neq 0$, then $x^2 > 0$.

Proof. If $x \neq 0$, then either x is negative or x is positive, but not both. If x is positive, then

$$x > 0,$$
$$(x)(x) > (0)(x), \qquad \text{(Why?)}$$
$$x^2 > 0.$$

If x is negative, then

$$x < 0$$

$$(x)(x) > (0)(x), \qquad \text{(Why?)}$$

$$x^2 > 0.$$

In either case, the result is the desired one.

The previous theorem states that the square of a non-zero number is positive. What can be said about x^2 for any x?

The properties of order can be used to advantage in finding truth sets of inequalities. For example, let us find the truth set of

$$(-3x) + 2 < 5x + (-6).$$

By the addition property of order we may add $(-2) + (-5x)$ to both members of this inequality to obtain

$$\Big((-3x) + 2\Big) + \Big((-2) + (-5x)\Big) < \Big(5x + (-6)\Big) + \Big((-2) + (-5x)\Big),$$

which when simplified is

$$-8x < -8.$$

Since $\Big((-2) + (-5x)\Big)$ is a real number for every value of x, the new sentence has the same truth set as the original. (What must we add to the members of "$-8x < -8$" to obtain the original sentence, that is, to reverse the step?)

Then, by the multiplication property of order.

$$(-8)(-\tfrac{1}{8}) < (-8x)(-\tfrac{1}{8})$$

$$1 < x$$

Here we multiplied by a non-zero real number. Thus, this sentence is equivalent to the former sentence. (What must we multiply the members of "$1 < x$" by to obtain the former sentence?) Obviously, the truth set of "$1 < x$" is the set of all numbers greater than 1, and this is the truth set of the original inequality.

Problems

1. Solve each of the following inequalities, using the form of the following example. (Recall that to "solve" a sentence is to find its truth set.)

   Example: $(-3x) + 4 < -5$.

   This sentence is equivalent to

   $-3x < (-5) + (-4)$, $\left(\text{add } (-4) \text{ to both members}\right)$

   $-3x < -9$,

   which is equivalent to

   $(-\frac{1}{3})(-9) < (-\frac{1}{3})(-3x)$, $\left(\text{multiply both members by } (-\frac{1}{3})\right)$

   $3 < x$.

   Thus, the truth set consists of all numbers greater than 3.

   (a) $(-2x) + 3 < -5$

   (b) $(-2) + (-4x) > -6$

   (c) $(-4) + 7 < (-2x) + (-5)$

   (d) $5 + (-2x) < 4x + (-3)$

   (e) $\frac{1}{2}x + (-2) < (-5) + \frac{5}{2}x$

   (f) $2x < 3 + |-2||-\frac{4}{3}|$

   (g) $-(2 + x) < 3 + (-7)$

2. Graph the truth sets of Parts (a) and (b) of Problem 1.

3. Translate the following into open sentences and solve.

   (a) Sue has 16 more books than Sally. Together they have more than 28 books. How many books does Sally have?

   (b) If a certain variety of bulbs are planted, less than $\frac{5}{8}$ of them will grow into plants. If, however, the bulbs are given proper care more than $\frac{3}{8}$ of them will grow. If a careful gardener has 15 plants, how many bulbs did he plant?

4. Prove that if $a < b$ and c is a negative number, then $bc < ac$. Hint: There is a negative number e such that $a = b + e$. Therefore, $ac = bc + ec$. What kind of number is ec? Hence, what is the order of ac and bc?

5. If $a < b$, and a and b are both positive real numbers, prove that $\frac{1}{b} < \frac{1}{a}$. Hint: Multiply the inequality $a < b$ by $(\frac{1}{a} \cdot \frac{1}{b})$. Demonstrate the theorem on the number line.

6. Does the relation $\frac{1}{b} < \frac{1}{a}$ hold if $a < b$ and both a and b are negative? Prove it or disprove it.

7. Does the relation $\frac{1}{b} < \frac{1}{a}$ hold if $a < b$ and $a < 0$ and $b > 0$? Prove or disprove.

8. State the addition and multiplication properties of the order ">".

9. Prove: If $0 < a < b$, then $a^2 < b^2$. Hint: Use properties of order to obtain $a^2 < ab$ and $ab < b^2$.

The Fundamental Properties of Real Numbers. We have been dealing with two main problems. The first problem was to extend the order relation and the operations of addition and multiplication from the numbers of arithmetic to all real numbers. Until this was done we really did not have the real number system to work with. The second problem was to discover and state carefully the fundamental properties of the real number system. The two problems, as we have been forced to deal with them, are closely intertwined. In this section we shall separate out the most important problem, the second one, by summarizing the fundamental properties which have been obtained.

Before continuing, we should admit that the decision as to what is a fundamental property is not made because of strict mathematical reasons but is to a large extent a matter of convenience and common agreement. We tend to think of the real number system and its many properties as a "structure" built upon a foundation consisting of fundamental properties. This is the way you should begin to think of the real number system. A good question, which can now be answered more precisely than before, is: What is the real number system?

The real number system is a set of elements for which binary operations of addition, "+", and multiplication, "·", along with an order relation, "<", are given with the following properties.

1. For any real numbers a and b,
   $a + b$ is a real number. (Closure)
2. For any real numbers a and b,
   $a + b = b + a$. (Commutativity)
3. For any real numbers a, b, and c,
   $(a + b) + c = a + (b + c)$. (Associativity)
4. There is a special real number 0
   such that, for any real number a,
   $a + 0 = a$. (Identity element)
5. For every real number a there is
   a real number $-a$ such that,
   $a + (-a) = 0$. (Inverses)
6. For any real numbers a and b,
   $a \cdot b$ is a real number. (Closure)
7. For any real numbers a and b,
   $a \cdot b = b \cdot a$. (Commutativity)
8. For any real numbers a, b, and c,
   $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. (Associativity)
9. There is a special real number 1
   such that, for any real number a,
   $a \cdot 1 = a$. (Identity element)

10. For any real number a different from 0, there is a real number $\frac{1}{a}$ such that,
$$a \cdot \left(\frac{1}{a}\right) = 1. \qquad \text{(Inverses)}$$

11. For any real numbers a, b, and c,
$$a \cdot (b + c) = a \cdot b + a \cdot c. \qquad \text{(Distributivity)}$$

12. For any real numbers a and b, exactly one of the following is true: $a < b$, $a = b$, $b < a$. (Comparison)

13. For any real numbers a, b, and c, if $a < b$ and $b < c$, then $a < c$. (Transitivity)

14. For any real numbers a, b, and c, if $a < b$, then $a + c < b + c$. (Addition property)

15. For any real numbers a, b, and c, if $a < b$ and $0 < c$, then
$$a \cdot c < b \cdot c,$$
if $a < b$ and $c < 0$, then
$$b \cdot c < a \cdot c. \qquad \text{(Multiplication property)}$$

You have probably noticed that there are several familiar and useful properties which we have failed to mention. This is not an oversight. The reasons for omitting them is that they can be proved from the properties listed here. In fact, by adding just one new property, we could obtain a list of properties from which everything about the real numbers could be proved. We shall not consider this additional property since that would take us beyond the limits of this course.

Practically all of the algebra in this course can be based on the above list of properties. It is by means of proofs that we bridge the gap between these basic properties and all of the many ideas and theorems which grow out of them. The chains of reasoning involved in proofs are what hold together the whole structure of mathematics -- or of any logical system.

Thus, if we are going to appreciate fully what mathematics is like, we should begin to examine how ideas are linked in these chains of reasoning -- we should do some proving and not always be satisfied with a plausible explanation. It is true that some of the statements we have proved seem very obvious, and you might wonder, quite justifiably, why we should bother to prove them. As we progress further in mathematics, there will be more ideas which are not at all obvious and which are established only through proofs. During the more elementary stages of our training we need the experience of seeing some simple proofs and developing gradually some feeling for the chain of reasoning on which the whole structure of mathematics depends. This is our reason for looking closely at proofs of some rather obvious statements.

The ability to discover a method for proving a theorem is something which does not develop overnight. It comes with seeing a variety of different proofs, by learning to look for connecting links between something you know and something you want to prove, by thinking about the suggestions which are given to lead you into a proof. On the other hand, the kind of thinking required is not used only in mathematics but is involved in all logical reasoning.

Let us now return to the fundamental properties of real numbers and summarize a few of the other properties which can be proved from those given on the previous page. Some of these were proved in the text and some were included in exercises.

16. Any real number x has just one additive inverse, namely, $-x$.

17. For any real numbers a and b,
$$-(a + b) = (-a) + (-b).$$

18. For real numbers a, b, and c, if $a + c = b + c$, then $a = b$.

19. For any real number a, $a \cdot 0 = 0$.

20. For any real number a, $(-1)a = -a$.

21. For any real numbers a and b, $(-a)b = -(ab)$ and $(-a)(-b) = ab$.

22. The opposite of the opposite of a real number a is a.

23. Any real number x different from 0 has just one multiplicative inverse, namely, $\frac{1}{x}$.

24. The number 0 has no reciprocal.

25. The reciprocal of a positive number is positive, and the reciprocal of a negative number is negative.

26. The reciprocal of the reciprocal of a non-zero real number a is a.

27. For any non-zero real numbers a and b,

$$\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}.$$

28. For real numbers a and b, $ab = 0$ if and only if $a = 0$ or $b = 0$.

29. For real numbers a, b, and c with $c \neq 0$, if $ac = bc$, then $a = b$.

30. For any real numbers a and b, if $a < b$, then $-b < -a$.

31. If a and b are real numbers such that $a < b$, then there is a positive number c such that $b = a + c$.

32. If $x \neq 0$, then $x^2 > 0$.

33. If $0 < a < b$, then $\frac{1}{b} < \frac{1}{a}$.

34. If $0 < a < b$, then $a^2 < b^2$.

You may have noticed that we have given a proof of the multiplication property of order in the preceding section. In fact, this property (No. 15 in the list) follows from the other 14 fundamental properties. Therefore, it could have been omitted from the list without limiting in any way its scope. However, we have included the property in order to emphasize the parallel between the properties of addition and the properties of multiplication.

You may have noticed also that nowhere in the previous discussion of fundamental properties is there any mention of absolute values. This important concept can be brought into the framework of the basic properties by the definition:

If $0 \leq a$, then $|a| = a$.
If $a < 0$, then $|a| = -a$.

We close this summary with a mention of some properties of a rather different kind, namely, the properties of equality. These are properties of the language of algebra rather than properties of real numbers. Recall that the sentence "$a = b$", where "a" and "b" are numerals, asserts that "a" and "b" name the same number. The first two properties of equality which we list have not been stated before but have actually been used many times. In the following, a, b, and c are any real numbers.

35. If $a = b$, then $b = a$. (Symmetry)
36. If $a = b$ and $b = c$, then $a = c$. (Transitivity)
37. If $a = b$, then $a + c = b + c$. (Addition property)
38. If $a = b$, then $ac = bc$. (Multiplication property)
39. If $a = b$, then $-a = -b$.
40. If $a = b$, then $|a| = |b|$.

FIELDS AND MODULO SYSTEMS

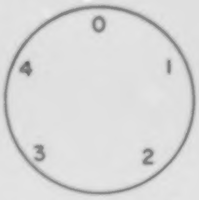
At this point, it might be worthwhile to "broaden our horizons a bit". Let us consider some of the properties we have just summarized, namely:

1. The real number system consists of a set of elements and two binary operations, addition and multiplication, such that:
2. This set of elements is closed under these operations. (Numbers 1 and 6.)
3. The operations of addition and multiplication are commutative. (Numbers 2 and 7.)
4. The operations of addition and multiplication are associative. (Numbers 3 and 8.)
5. The operations satisfy the distributive property which states that $a(b + c) = a \cdot b + a \cdot c$. (Number 11.)
6. There are distinct identity elements for addition and multiplication, namely, 0 and 1, respectively, for the real numbers. (Numbers 4 and 19.)
7. For each element there exists an additive inverse, such that the sum of any element and its additive inverse equals the identity element for addition. (Number 4.)
8. For each element other than 0 there exists a multiplicative inverse, such that the product of any non-zero element and its multiplicative inverse equals the identity element for multiplication.

Mathematicians call such a system, which satisfies these properties, a field. The real number system satisfies these properties and is obviously a field. This is not the only system which is a field, however. The set of rational numbers with addition and multiplication also is a field, as is the complex number system, to name a few. It should be noted that the sets in these systems are infinite; i.e., there is no limit to the number of elements in the set.

The question arises as to the possibility of constructing a field with a finite set of elements.

Let us consider the system consisting of the finite set of integers {1, 2, 3, 4, 0} and the operations of "addition" and "multiplication" defined as follows. Imagine a clock with these numbers on its face as follows:



We see that one unit more than 2 is 3, or $2 + 1 = 3$, and that $1 + 3 = 4$. However, what is $3 + 4$ in this system? Clearly, it is 4 units more than 3, or 2; hence, $3 + 4 = 2$ also, $1 \cdot 3 = 3$, $2 \cdot 3 = 1$, etc. We call such a system, "integers modulo 5", which means that we can count to 4 and must start all over again. Our usual system of telling time is a modulo system. How would you describe it?

Let us proceed to construct addition and multiplication tables, remembering that we will be working in the modulo 5 system.

One table will look like this to start,

+	0	1	2	3	4
0					
1					
2		3			
3					
4					

and will read something like a mileage chart on a road map. To find $2 + 1$ we would locate the row containing 2 and the column containing 1 and notice where they cross. To complete the table we may have to refer back to our clock, but when finished it will look like this:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Now, does $4 + 2 = 2 + 4$? Is this "addition" commutative? Is there an identity element for addition? How many other field properties of "addition" hold in this system?

There follows a partially constructed table for the operation of multiplication $(\cdot)$ in this system. The reader is to fill in the blanks.

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	__	__
4	0	__	__	__	1

Before attempting to complete the table the following discussion might be helpful. Let us consider the following products!

$0 = 0 \cdot 5 + 0$
$1 = 0 \cdot 5 + 1$
$2 = 0 \cdot 5 + 2$
$3 = 0 \cdot 5 + 3$
$4 = 0 \cdot 5 + 4$
$5 = 1 \cdot 5 + 0$
$6 = 1 \cdot 5 + 1$
$7 = 1 \cdot 5 + 2$
$8 = 1 \cdot 5 + 3$

$9 = 1 \cdot 5 + 4$
$10 = 2 \cdot 5 + 0$
$11 = 2 \cdot 5 + 1$
$12 = 2 \cdot 5 + 2$
$13 = 2 \cdot 5 + 3$
$14 = 2 \cdot 5 + 4$
$15 = 3 \cdot 5 + 0$
$16 = 3 \cdot 5 + 1$
etc...

By combing this information and the numbers in the table, it is seen that the products modulo 5 are nothing more than the remainders when division of the ordinary products has been performed by 5 or a multiple of 5. For example, $2 \cdot 4 = 8$ but $8 = 1 \cdot 5 + 3$, hence, we see that $2 \cdot 4 = 3 \pmod 5$.

Although we have used this idea for multiplication it may be used just as easily to construct the addition table in modulo 5 arithmetic. All that must be done is to divide the ordinary sums by 5 or a multiple of 5. The remainder is the corresponding entry of the addition table. The reader should verify this with the addition table on page 174.

Now to answer our basic question, "Is the system of integers modulo 5 a field?" To determine this, we must see that all of the 8 properties of a field are satisfied.

From the tables it is fairly obvious that the closure, commutative, associative and distributive properties are satisfied. We also see that the identity elements 0 and 1 appear. Further examination is necessary to verify if there are such things as additive inverses. We see that $1 + 4 = 0$, $2 + 3 = 0$, $3 + 2 = 0$ and $4 + 1 = 0$. Hence, the additive inverse of 1 must be 4, of 2 must be 3, and so on. In a like manner we can check and see that every element (except 0) has one and only one multiplicative inverse. ($2 \cdot 3 = 1$, for example.)

Evidently the system of integers modulo 5 is a field! Notice, however, that we must be very careful concerning the order relations between the elements of this set. In fact, we shall find that modulo systems such as this do not satisfy all of the properties of order. (Numbers 12-15 on page 168.)

Is it safe to assume that all modulo systems are fields? It is neither safe nor wise to do so, for not all such systems are fields.

Some modulo systems will satisfy all of the field properties except one, (the same one) and, hence, they cannot qualify for membership in the "fields" club.

Problems

1. As was done in the text, construct tables for addition and multiplication for the following systems of integers!

   (a) modulo 2 (c) modulo 4

   (b) modulo 3 (d) modulo 6

2. Which of the systems in Problem 1 are fields?

3. What is the only field property that a modulo system does not satisfy if it is not a field?

Part 5

ADDITIVE AND MULTIPLICATIVE INVERSES

Definition of Subtraction. Suppose you make a purchase which amounts to 83 cents, and give the cashier one dollar. What does she do? She puts down two cents and says "85", one nickel and says "90", and one dime and says "one dollar". What has she been doing? She has been subtracting 83 from 100. How does she do it? - by finding what she has to *add* to 83 to obtain 100. The question "100 - 83 = what?" means the same as "83 + what = 100?". And how have we solved the equation

$$83 + x = 100$$

so far in this course? We add the opposite of 83, and find

$$x = 100 + (-83).$$

Thus, "100 - 83" and "100 + (-83)" are names for the same number.

Try a few more examples:

$20 - 9 = 11$	$20 + (-9) = 11$
$8 - 6 = 2$	$8 + (\ \ ) = 2$
$5 - 2 = (\ )$	$5 + (\ \ ) = 3$
$8.5 - (\ ) = 5.3$	$8.5 + (-3.2) = (\ ).$

From these examples you will agree that subtracting a positive number b from a larger positive number a, gives the same result as adding the opposite of b to a.

Our problem now is to decide how to define subtraction for *all* real numbers. We have now described subtraction in the familiar case of the positive numbers in terms of operations we know how to do for all real numbers, namely, adding and taking opposites. And so we *define* subtraction for all real numbers as adding the opposite. In this way, we *extend* subtraction to real numbers so that it still has the properties we know from arithmetic; and our definition has used only ideas with which we have previously become familiar.

To *subtract* the real number b from the real number a, add the opposite of b to a. Thus, for real numbers a and b,

$$a - b = a + (-b).$$

Examples:

$$\begin{aligned} 2 - 5 &= 2 + (-5) = -3 \\ 5 - 2 &= 5 + (-2) = \ \ 3 \\ (-2) - 5 &= (-2) + (-5) = -7 \\ 2 - (-5) &= 2 + 5 = 7 \\ (-5) - 2 &= ? \\ 5 - (-2) &= ? \\ (-2) - (-5) &= ? \\ (-5) - (-2) &= ? \end{aligned}$$

Read the expression "5 - (-2)". Is the symbol "-" used in two different ways? What is the meaning of the first "-"? What is the meaning of the second "-"?

To help keep these uses of the symbol clear, we make the following parallel statements about them.

In "a - b",	In "a + (-b)"
"-" stands *between* two numerals and indicates the operation of *subtraction*. We read the above as "a minus b".	"-" is part of one numeral and indicates the *opposite of*. We read the above as "a plus the opposite of b".

We see that the operation of subtraction is closely related to that of addition. We may state this as

Theorem. For any real numbers a, b, c,

$$a = b + c \text{ if and only if } a - b = c.$$

Proof. Remember that in order to prove a theorem involving "if and only if" we really must prove two theorems.

Let us first prove: if $a = b + c$, then $a - b = c$.

$$\begin{aligned} a &= b + c \\ a + (-b) &= (b + c) + (-b) && \text{(Why?)} \\ a - b &= \big(b + (-b)\big) + c && \text{(Why?)} \\ a - b &= c. && \text{(Why?)} \end{aligned}$$

Next we prove: If $a - b = c$, then $a = b + c$. To do this, note that "$a - b = c$" means "$a + (-b) = c$". The reader may now complete the proof.

Problems

1. $\frac{3}{4} - (-\frac{1}{2})$

3. $(-1.79) - 1.22$

2. $(-0.631) - (0.631)$

4. $75 - (-85)$

5. Subtract -8 from 15.

6. From -25, subtract -4.

7. How much greater is 8 than -5?

8. Let R be the set of all real numbers, and S the set of all numbers obtained by performing the operation of subtraction on pairs of numbers of R. Is S a subset of R? Are the real numbers closed under subtraction? Are the numbers of arithmetic closed under subtraction?

9. Find the truth set of each of the following equations:

   (a) $z - 34 = 76$

   (c) $z + (-\frac{3}{4}) = -\frac{1}{2}$

   (b) $2x + 8 = -16$

10. From a termperature of 3° below zero, the temperature dropped 10°. What was the new termperature? Show how this question is related to subtraction of real numbers.

11. The bottom of Death Valley is 282 feet below sea level. The top of Mt. Whitney, which is visible from Death Valley, has an altitude of 14,495 feet above sea level. How high above Death Valley is Mt. Whitney?

Properties of Sbutraction. What are some of the properties of subtraction? Is

$$5 - 2 = 2 - 5?$$

What do you conclude about the commutativity of subtraction? Next, is

$$8 - (7 - 2) = (8 - 7) - 2?$$

Do you think subtraction is associative?

If subtraction does not have some of the properties to which we have become accustomed, we shall have to learn to subtract by going back to the definition in terms of adding the opposite. Addition, after all, does have the familiar properties.

For example, since subtraction is not associative, the expression

$$3 - 2 - 4$$

really is not a numeral because it does not name a specific number. Recall that subtraction is a binary operation, that is, involves two numbers. Then does "3 - 2 - 4" mean "3 - (2 - 4)" or does it mean "(3 - 2) - 4"? To make a decision, we convert subtraction to addition of opposite. Then

$$\begin{aligned} 3 - (2 - 4) &= 3 + \Big(-\big(2 + (-4)\big)\Big) \\ &= 3 + \big((-2) + 4\big) \\ &= 3 + (-2) + 4 \end{aligned}$$

On the other hand,

$$\begin{aligned} (3 - 2) - 4 &= \big(3 + (-2)\big) + (-4) \\ &= 3 + (-2) + (-4). \end{aligned}$$

The second of these is the meaning we decide upon. We shall agree that

$$a - b - c \text{ means } (a - b) - c,$$

that is,

$$a - b - c = a + (-b) + (-c).$$

Example 1. Find a common name for

$$\left(\frac{6}{5} + 2\right) - \frac{1}{5}.$$

We can think of this as $\left(\frac{6}{5} + 2\right) + \left(-\frac{1}{5}\right)$, and then we know that we can write

$$\begin{aligned}\left(\frac{6}{5} + 2\right) - \frac{1}{5} &= \left(\frac{6}{5} + 2\right) + \left(-\frac{1}{5}\right)\\ &= \left(\frac{6}{5} + \left(-\frac{1}{5}\right)\right) + 2\\ &= 1 + 2\\ &= 3.\end{aligned}$$

Example 2. Use the properties of addition to write $-3x + 5x - 8x$ in simpler form.

$$-3x + 5x - 8x = (-3)x + 5x + (-8)x,$$

where we have used the theorem, $-ab = (-a)b$, for the first term, and the definition of subtraction and the same theorem for the last term. That is, we think of $-3x + 5x - 8x$ as the sum of $(-3)x$, $5x$, and $(-8)x$. Then

$$\begin{aligned}-3x + 5x - 8x &= \big((-3) + 5 + (-8)\big)x \quad \text{by the distributive property}\\ &= -6x.\end{aligned}$$

While it is not as precise, we use the commonly accepted word "simplify" for directions such as "find a common name for" and "use the properties of addition to write the following in simpler form". When there is no possibility of confusion, this term will appear henceforth.

Example 3. Simplify $(5y - 3) - (6y - 8)$.

$$\begin{aligned}(5y - 3) - (6y - 8) &= 5y + (-3) + \big(-(6y + (-8))\big) && \text{(Why?)}\\ &= 5y + (-3) + \big(-(6y)\big) + \big(-(-8)\big), && \text{since the opposite of the sum is the sum of the opposites.}\\ &= 5y + (-3) + (-6)y + 8 && \text{(Why?)}\\ &= (-1)y + 5 && \text{(Why?)}\\ &= -y + 5.\end{aligned}$$

Instead of the fact that the opposite of a sum is the sum of the opposites, we could also have used the theorem which states that $-a = (-1)a$, and then the distributive property. Then our example would have proceeded as follows:

$$\begin{aligned}(5y - 3) - (6y - 8) &= 5y - 3 + -(6y - 8)\\ &= 5y - 3 + (-1)\big(6y + (-8)\big)\\ &= 5y - 3 + (-1)(6y) + (-1)(-8)\\ &= 5y - 3 - 6y + 8\\ &= -y + 5\end{aligned}$$

When you understand the steps involved, you can abbreviate the steps to:

$$\begin{aligned}(5y - 3) - (6y - 8) &= 5y - 3 - 6y + 8\\ &= -y + 5\end{aligned}$$

You may be impressed by the way we are now doing a number of steps mentally. This ability to comprehend several steps without writing them all down is a sign of our mathematical growth. We must be careful, however, to be able at any time to pick out all the detailed steps and explain each one.

Problems

Simplify:

1. (a) $-3y - 5y + y$ (c) $3a^2 - 5a^2 + 6a$
   (b) $-3c + 5c - \frac{1}{2}c$
2. $-(3x - 4y)$
3. $(\frac{3}{4} - \frac{1}{6}) + (\frac{1}{4} + \frac{1}{6})$
4. $(3x - 6) + (7 - 4x)$
5. $(5a - 17b) - (4a - 6b)$
6. $(2x + 7) + (4x^2 + 8 - x)$
7. $(3a + 2b - 4) - (5a - 3b + c)$

8. $(5x - 3y) - (2 + 5x) + (3y - 2)$

9. What must be added to $3s - 4t + 7u$ to obtain $-9s - 3u$?

10. Prove: If $a > b$, then $a - b$ is positive.

11. If $(a - b)$ is a positive number, which of the statements, $a < b$, $a = b$, $a > b$, is true? What if $(a - b)$ is a negative number? What if $(a - b)$ is zero?

12. If a, b, and c are real numbers and $a > b$, what can we say about the order of $a - c$ and $b - c$? Prove your statement.

The definition of subtraction in terms of addition permits us to extend further our applications of the distributive property, and to describe in different language some of our steps in finding truth sets.

Example. Simplify

$$(-3)(2x - 5).$$

By applying the definition of subtraction, we have

$$\begin{aligned}(-3)(2x - 5) &= (-3)\big(2x + (-5)\big)\\ &= (-3)(2x) + (-3)(-5) \qquad \text{(Why?)}\\ &= \big((-3)(2)\big)x + 15 \qquad \text{What properties of multiplication have we used here}\\ &= (-6)x + 15\\ &= -6x + 15.\end{aligned}$$

You would perhaps have done some of these steps mentally, and would have written directly:

$$(-3)(2x - 5) = -6x + 15,$$

thinking "(-3) times (2x) is (-6x)"
"(-3) times (-5) is 15."

Problems

1. Perform indicated operations and simplify where possible:

    (a) $(-3)(-a + 2b - c)$

    (b) $(-3x + 2y) + 2(-2x - y)$

    (c) $(-2)(a - 2b) + 3(a - 2b)$

    (d) $3(a - b + c) - (2a - b - 2c)$

    (e) $a(b + c + 1) - 2a(2b + c - 1)$

2. Solve:

    (a) $2a - 1 = 4a - 3$

    (b) $-3y = 2 - y - 6$

    (c) $-2 - 2y < -1$

    (d) $4u + 3 > -5u - 3$

    (e) $3a + 3 = 7a + 4 - 4a - 1$

3. (a) The width of a rectangle is 5 inches less than its length. What is its length if its perimeter is 38 inches?

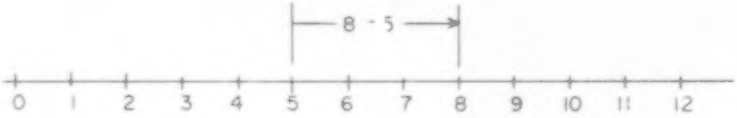
    (b) If 17 is subtracted from a number, and the result is multiplied by 3, the product is 102. What is the number?

    (c) A teacher says, "If I had 3 times as many students in my class as I do have, I would have less than 46 more than I now have." How many students does he have in his class?

Subtraction in Terms of Distance. Suppose we ask: On the number line, how far is it from 5 to 8? If x represents the number of units in this distance, then

$$5 + x = 8.$$

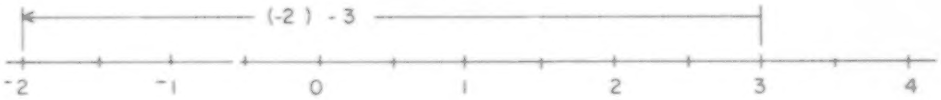
The solution of this equation, as we have seen, can be written as $x = 8 - 5$. Thus, $8 - 5$ can be interpreted as the distance from 5 to 8 on the number line.



Let us now ask how far it is from 3 to (-2). If y represents the number of units in this distance, then

$$3 + y = (-2)$$
$$y = (-2) - 3.$$

Thus, $(-2) - 3$ can be interpreted as the distance from 3 to (-2).



The quantity $8 - 5 = 3$ is positive, while $(-2) - 3$ is negative. What does this distinction tell us? It tells us that the distance from 5 to 8 is to the *right*, while from 3 to (-2) is to the *left*. Therefore, $a - b$ really gives us the distance *from* b *to* a, that is, both the length and its direction.

Suppose we are not interested in the direction, but only in the distance *between* a and b. Then $a - b$ is the distance from b to a, and $b - a$ is the distance from a to b, and the distance *between* a and b is the positive of these two. From our earlier work, we know that this is $|a - b|$.

For example, the distance from 3 to (-2) is $(-2) - 3$, that is, -5; the distance between 3 and (-2) is $|(-2) - 3|$, that is, 5. In the same way, the distance from 2 to x is $x - 2$; the distance between 2 and x is $|x - 2|$.

Problems

1. What is the distance
   (a) from -3 to 5?
   (b) between -3 and 5?
   (c) from 6 to -2?
   (d) between 6 and -2?
   (e) from 5 to 1?
   (f) between 5 and 1?
   (g) from -8 to -2?
   (h) between -8 and -2?
   (i) from 7 to 0?
   (j) between 7 and 0?
2. What is the distance
   (a) from x to 5?
   (b) between x and 5?
   (c) from -2 to x?
   (d) between -2 and x?
   (e) from -1 to $-x$?
   (f) between $-x$ and -1?
   (g) from 0 to x?
   (h) between 0 and x?
3. For each of the following pairs of expressions, fill in the symbols "<", "=", or ">", which will make a true sentence.
   (a) $|9 - 2|$? $|9| - |2|$
   (b) $|2 - 9|$? $|2| - |9|$
   (c) $|9 - (-2)|$? $|9| - |-2|$
   (d) $|(-2) - 9|$? $|-2| - |9|$
   (e) $|(-9) - 2|$? $|-9| - |2|$
   (f) $|2 - (-9)|$? $|2| - |-9|$
   (g) $|(-9) - (-2)|$? $|-9| - |-2|$
   (h) $|(-2) - (-9)|$? $|-2| - |-9|$

4. Write a symbol between $|a - b|$ and $|a| - |b|$ which will make a true sentence for all real numbers a and b. Do the same for $|a - b|$ and $|b| - |a|$. For $|a - b|$ and $||a| - |b||$.

5. What are the two numbers x on the number line such that

$$|x - 4| = 1?$$

6. What is the truth set of the sentence

$$|x - 4| < 1?$$

Draw the graph of this set on the number line.

7. What is the truth set of the sentence

$$|x - 4| > 1?$$

8. Graph the truth set of

$$x > 3 \text{ and } x < 5$$

on the number line. Is this set the same as the truth set of $|x - 4| < 1$? (We usually write "$3 < x < 5$" for the sentence "$x > 3$ and $x < 5$".)

9. Find the truth set of each of the following equations; graph each of these sets:

(a) $|x - 6| = 8$

(b) $y + |-6| = 10$

(c) $|10 - a| = 2$

(d) $|x| < 3$

(e) $|v| > -3$

(f) $|y| + 12 = 13$

(g) $|y - 8| < 4$ (Read this: The distance between y and 8 is less than 4.)

(h) $|z| + 12 = 6$

(i) $|x - (-19)| = 3$

(j) $|y + 5| = 9$

10. For each sentence in the left column pick the sentence in the right column which has the same truth set:

$|x| = 3$
$|x| < 3$
$|x| \leq 3$
$|x| > 3$
$|x| \ngtr 3$

$x = -3$ and $x = 3$
$x = -3$ or $x = 3$
$x > -3$ and $x < 3$
$x > -3$ or $x < 3$
$x < -3$ and $x > 3$
$x < -3$ or $x > 3$
$x \geq -3$ and $x \leq 3$

11. From a point marked 0 on a straight road, John and Rudy ride bicycles. John rides 10 miles per hour and Rudy rides 12 miles per hour. Find the distance between them after

(1) 3 hours, (2) $1\frac{1}{2}$ hours, (3) 20 minutes, if

(a) They start from the 0 mark at the same time and John goes east and Rudy goes west.

(b) John is 5 miles east and Rudy is 6 miles west of the 0 mark when they start and they both go east.

(c) John starts from the 0 mark and goes east. Rudy starts from the 0 mark 15 minutes later and goes west.

(d) Both start at the same time. John starts from the 0 mark and goes west and Rudy starts 6 miles west of the 0 mark and also goes west.

Division. You will recall that we defined subtraction of a number as addition of the opposite of the number:

$$a - b = a + (-b).$$

In other words, we defined subtraction in terms of addition and the additive inverse.

Since division is related to multiplication in much the same way as subtraction is related to addition, we might expect to define division in terms of multiplication and the multiplicative inverse. This is exactly what we do.

> For any real numbers a and b $(b \neq 0)$, "a divided by b" means "a multiplied by the reciprocal of b".

We shall indicate "a divided by b" by the symbol "$\frac{a}{b}$". This symbol is not new. You have used it as a fraction indicating division. Then the definition of division is:

$$\frac{a}{b} = a \cdot \frac{1}{b}, \quad (b \neq 0).$$

As in arithmetic, we shall call "a" the numerator and "b" the denominator of the fraction "$\frac{a}{b}$". When there is no possibility of confusion, we shall also call the number named by "a" the numerator, and the number named by "b" the denominator.

Here are some examples of our definition. By $\frac{10}{2}$, we mean $10 \cdot \frac{1}{2}$, or 5; by $\frac{3}{\frac{1}{5}}$ we mean $3\left(\frac{1}{\frac{1}{5}}\right) = 3(5) = 15$.

Does this definition of division agree with the ideas about division which we already have in arithmetic? An elementary way to talk about $\frac{10}{2}$ is to ask "what times 2 gives 10"? Since $5 \cdot 2 = 10$, then $\frac{10}{2} = 5$.

Why in the definition of division did we make the restriction "$b \neq 0$"? Be on your guard against being forced into an impossible situation by inadvertently trying to divide by zero.

Problems

Write common names for the following:

1. $\frac{-30}{-5}$
2. $\frac{30}{-5}$
3. $\frac{-30}{5}$
4. $\frac{4}{\frac{3}{5}}$
5. $\frac{\frac{2}{3}}{\frac{1}{6}}$
6. $\frac{\frac{7}{8}}{2}$
7. $\frac{2b}{b}$

Is the following theorem consistent with your experience in arithmetic?

Theorem. For $b \neq 0$, $a = cb$
if and only if $\frac{a}{b} = c$.

This amounts to saying that a divided by b is the number which multiplied by b gives a. Compare this with the theorem which says that b subtracted from a is the number which added to b gives a.

Again, in order to prove a theorem involving "if and only if" we must prove two things. First, we must show that if $\frac{a}{b} = c$ $(b \neq 0)$, then $a = cb$. The fact that we want to obtain cb on the right suggests starting the proof by multiplying both members of "$\frac{a}{b} = c$" by b.

Proof. If $\frac{a}{b} = c$ $(b \neq 0)$, then $a \cdot \frac{1}{b} = c$,

$$\begin{aligned}(a \cdot \tfrac{1}{b})b &= cb,\\ a(\tfrac{1}{b} \cdot b) &= cb,\\ a \cdot 1 &= cb,\\ a &= cb.\end{aligned}$$

Second, we must show that if $a = cb$ $(b \neq 0)$, then $\frac{a}{b} = c$. This time, the fact that we do not want b on the right suggests starting the proof by multiplying both members of "$a = cb$" by $\frac{1}{b}$. This is possible, since $b \neq 0$.

Proof. If $a = cb$ $(b \neq 0)$, then

$$\begin{aligned} a\cdot\frac{1}{b} &= (cb)\frac{1}{b}, \\ a\cdot\frac{1}{b} &= c\left(b\cdot\frac{1}{b}\right), \\ a\cdot\frac{1}{b} &= c\cdot 1, \\ a\cdot\frac{1}{b} &= c, \\ \frac{a}{b} &= c. \end{aligned}$$

Supply the reason for each step of the above proofs.

The second part of this theorem agrees with our customary method of checking division by multiplying the quotient by the divisor.

The multiplication property of 1 states that $a = a(1)$ for any real number a. If we apply this theorem here, we obtain two familiar special cases of division. For any real number a,

$$\frac{a}{1} = a,$$

and for any non-zero real number a,

$$\frac{a}{a} = 1.$$

Problems

1. Prove that for any real number a,

$$\frac{a}{1} = a.$$

2. Prove that for any non-zero real number a,

$$\frac{a}{a} = 1.$$

3. In the following problems perform the indicated divisions and check by multiplying the quotient by the divisor.

   (a) $\frac{-45}{5}$

   (b) $\frac{-200}{-50}$

   (c) $\frac{3\sqrt{5}}{3}$

   (d) $\frac{-\frac{2}{3}}{3}$

   (e) $\frac{12}{\frac{1}{3}}$

   (f) $\frac{0}{40}$

   (g) $\frac{28}{0}$

4. When dividing a positive number by a negative number, is the quotient positive or is it negative? What if we divide a negative number by a positive number? What if we divide a negative number by a negative number?

5. Find the truth set of each of the following equations:

   (a) $6y = 42$

   (b) $-6y = 42$

   (c) $6y = -42$

   (d) $-6y = -42$

   (e) $\frac{1}{5}x = 20$

   (f) $\frac{x}{5} = 15$

   (g) $\frac{3}{4}a = 9$

   (h) $\frac{2}{3}b = 0$

   (i) $5x = \frac{10}{3}$

6. Find the truth set of each of the following equations.

   (a) $5a - 8 = -53$

   (b) $\frac{3}{4}y + 13 = 25$

   (c) $x + .30x = 6.50$

7. If six times a number is decreased by 5, the result is -37. Find the number.

8. If two-thirds of a number is added to 32, the result is 38. What is the number?

9. John is three times as old as Dick. Three years ago the sum of their ages was 22 years. How old is each now?

10. Find two consecutive even integers whose sum is 46.

11. Find two consecutive odd positive integers whose sum is less than or equal to 83.

12. Two trains leave Chicago at the same time: one travels north at 60 m.p.h. and the other south at 40 m.p.h. After how many hours will they be 125 miles apart?

13. One-half of a number is 3 more than one-sixth of the same number. What is the number.

14. John has 50 coins which are nickels, pennies, and dimes. He has four more dimes than pennies, and six more nickels than dimes. How many of each kind of coin has he? How much money does he have?

15. John, who is saving his money for a bicycle, said, "In five weeks I shall have one dollar more than three times the amount I now have. I shall then have enough money for my bicycle." If the bicycle costs $76, how much money does John have now?

16. The sum of two successive positive integers is less than 25. Find the integers.

17. A syrup manufacturer made 160 gallons of syrup worth $608 by mixing maple syrup worth $2 per quart with corn syrup worth 60 cents per quart. How many gallons of each kind did he use?

18. Show that if the quotient of two real numbers is positive, the product of the numbers also is positive, and if the quotient is negative, the product is negative.

Common Names. In Part 1 we noted some special names for rational numbers which are in some sense the simplest names for these numbers, and which we called "common names". Two particular items of interest about indicated quotients were the following: We do not call "$\frac{20}{5}$" a common name for "four", because "4" is simpler; similarly, "$\frac{14}{21}$" is not a common name for "two-thirds", because "$\frac{2}{3}$" is simpler. We obtain these common names by using the property of 1 and the theorem $\frac{a}{a} = 1$.

$$\frac{20}{5} = \frac{4\cdot 5}{5} = 4\left(\frac{5}{5}\right) = 4(1) = 4$$

and

$$\frac{14}{21} = \frac{2\cdot 7}{3\cdot 7} = \frac{2}{3}\left(\frac{7}{7}\right) = \frac{2}{3}(1) = \frac{2}{3}.$$

On the other hand, we cannot simplify "4" and "$\frac{2}{3}$" any further.

In the above example, what permitted us to write $\frac{2\cdot 7}{3\cdot 7}$ as $\frac{2}{3}\left(\frac{7}{7}\right)$? This familiar practice from arithmetic is one which can be proved for all real numbers.

Theorem. For any real numbers a, b, c, d, if $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd}.$$

Proof.

$$\frac{a}{b}\cdot\frac{c}{d} = \left(a\cdot\frac{1}{b}\right)\left(c\cdot\frac{1}{d}\right) \qquad \text{(Why?)}$$

$$= (ac)\left(\frac{1}{b}\cdot\frac{1}{d}\right) \qquad \text{(Why?)}$$

$$= (ac)\left(\frac{1}{bd}\right) \qquad \text{(Why?)}$$

$$= \frac{ac}{bd} \qquad \text{(Why?)}$$

Example 1. Simplify $\frac{3a^2b}{5aby}$.

$$\frac{3a^2b}{5aby} = \frac{3a(ab)}{5y(ab)}, \quad \text{by associative and commutative properties,}$$

$$= \frac{3a}{5y}\left(\frac{ab}{ab}\right), \quad \text{(Why?)}$$

$$= \frac{3a}{5y}, \quad \text{by the property of } 1.$$

Example 2. Simplify $\frac{3y - 3}{2(y - 1)}$. (Note: When we write this phrase, we assume automatically that the domain of y excludes 1. Why?)

$$\frac{3y - 3}{2(y - 1)} = \frac{3(y - 1)}{2(y - 1}, \quad \text{by the distributive property,}$$

$$= \frac{3}{2}\left(\frac{y - 1}{y - 1}\right), \quad \text{(Why?)}$$

$$= \frac{3}{2}(1), \quad \text{since } \frac{a}{a} = 1, \text{ (here } a = y - 1)$$

$$= \frac{3}{2}, \quad \text{by the multiplication property of } 1.$$

After further experience, your mental agility will undoubtedly permit you to skip some of these steps.

Example 3. Simplify $\frac{(2x + 5) - (5 - 2x)}{-8}$.

By the definition of subtraction,

$$\frac{(2x + 5) - (5 - 2x)}{-8} = \frac{2x + 5 - 5 + 2x}{-8}$$

$$= \frac{4x}{-8},$$

$$= \frac{x(4)}{-2(4)},$$

$$= \frac{x}{-2} \quad \text{by the multiplication property of } 1.$$

The numerals $\frac{x}{-2}$ and $\frac{-x}{2}$ and $-\frac{x}{2}$ all name the same number, and all look equally simple; the accepted common name is the last of these. Therefore,

$$\frac{2x + 5 - (5 - 2x)}{-8} = -\frac{x}{2}.$$

Problems

1. We have used the property of real numbers a and b that if $b \neq 0$, then

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}.$$

Prove this theorem.

In each of the following problems, simplify:

2. (a) $\frac{12}{9}$ (b) $\frac{-12}{10 - 1}$

3. (a) $\frac{-n^2}{n}$ (b) $\frac{-n}{-n^2}$

4. (a) $\frac{2(x - 2)}{3(x - 2)}$ (c) $\frac{2(x - 2)}{3(2 - x)}$

(b) $\frac{2x - 4}{3x - 6}$

5. (a) $\frac{xy + y}{x + 1}$ (c) $\frac{xy + y}{y}$

(b) $\frac{xy - y}{x - 1}$ (d) $\frac{xy - y}{y(x - 1)}$

6. (a) $\frac{8b - 10}{4b - 5}$ (c) $\frac{8(1 - b) + 2}{4b - 5}$

(b) $\frac{8b - 10}{5 - 4b}$

7. (a) $\frac{x + 2}{3}$ (c) $\frac{2x + 1}{3}$

(b) $\frac{2x - 3}{2y - 3}$ (d) $\frac{3x + 6}{3}$

8. (a) $\frac{2a - a^2}{a}$ (c) $\frac{2a - a^2}{a^2 - 2a}$

(b) $\frac{2a - a^2}{a - 2}$

9. (a) $\frac{(x + 1)(x - 1)}{x + 1}$ (c) $\frac{(x - 1)(2x - 3 + x)}{4x - 4}$

(b) $\frac{(x + 1)(x - 1)}{4x - 4}$ (d) $\frac{(-5x - 5)(2 - 2x)}{10x + 10}$

Fractions. Let us recall two conventions on common names which we have been using ever since Part 1: A common name contains no indicated division which can be performed, and if it contains an indicated division, the resulting fraction should be "in lowest terms". Then in this part, we stated another convention, this one about opposites: We prefer writing $-\frac{a}{b}$ to any of the other simple names for the same number, $\frac{-a}{b}$, $\frac{a}{-b}$.

Let us return to the conventions about fractions. In this course a "fraction" is a symbol which indicates the quotient of two numbers. Thus, a fraction involves two numerals, a numerator and denominator. When there is no possibility of confusion, we shall use the word "fraction" to refer also to the number itself which is represented by the fraction. When there is a possibility of confusion we must go back to our strict meaning of fraction as a numeral.

In some applications of mathematics the number given by $\frac{a}{b}$ is called the ratio of a to b. Again we shall sometimes speak of the ratio when we mean the symbol indicating the quotient.

Example 1. Simplify $\frac{x}{3}\cdot\frac{5}{6}$.

$$\frac{x}{3}\cdot\frac{5}{6} = \frac{x\cdot 5}{3\cdot 6} \quad \text{(What theorem is used here?)}$$

$$= \frac{5x}{18}.$$

Example 2. Simplify $(\frac{3}{2})(\frac{14}{9})$.

$$
\begin{aligned}
(\tfrac{3}{2})(\tfrac{14}{9}) &= \frac{3\cdot 14}{2\cdot 9} && \text{(Why?)}\\
&= \frac{3\cdot(2\cdot 7)}{2\cdot(3\cdot 3)} && \text{because } 14 = 2\cdot 7 \text{ and } 9 = 3\cdot 3.\\
&= \frac{7\cdot(2\cdot 3)}{3\cdot(2\cdot 3)} && \text{(Why?)}\\
&= \frac{7}{3}\left(\frac{2\cdot 3}{2\cdot 3}\right) && \text{(Why?)}\\
&= \frac{7}{3} && \text{by the property of } 1.
\end{aligned}
$$

Problems

In Problems 1-8 simplify:

1. $\frac{3}{8}\cdot\frac{7}{2}$

2. (a) $\frac{4}{7}\cdot\frac{21}{10}$ (b) $\frac{4}{5+2}\cdot\frac{7+14}{7}$

3. (a) $(-2)\cdot\frac{5}{9}$ (b) $\frac{1}{9}\cdot\big((-5)(-2)\big)$

4. (a) $\frac{x}{4}\cdot\frac{x}{3}$ (b) $\frac{x}{4}\cdot\frac{3}{x}$

5. (a) $\frac{10}{3}\cdot\frac{3}{2}$ (c) $(-\frac{10}{3})(-\frac{3}{2})$

(b) $\frac{10}{3}+\frac{3}{2}$ (d) $(-\frac{10}{3})+(-\frac{3}{2})$

6. (a) $\frac{3}{4}\cdot\frac{x+2}{3}$ (b) $3\cdot\frac{(x+2)}{4}$

7. (a) $\frac{n+3}{2}\cdot\frac{n+2}{3}$ (c) $\frac{n+3}{n+2}\cdot\frac{2}{3}$

(b) $\frac{n+3}{2}\cdot\frac{2}{n+3}$

8. (a) $\frac{xy+y}{x+1}\cdot\frac{xy-y}{x-1}$ (b) $\frac{2a-a^2}{-a}\cdot\frac{2a}{a-2}$

9. Can every rational number be represented by a fraction? Does every fraction represent a rational number?

10. The ratio of faculty to students in a college is $\frac{2}{19}$. If there are 1197 students, how many faculty members are there?

11. The profits from a student show are to be given to two scholarship funds in the ratio $\frac{2}{3}$. If the fund receiving the larger amount was given \$387, how much was given to the other fund?

We can state what we have done so far in another way. A product of two indicated quotients can always be written as one indicated quotient. Thus, in certain kinds of phrases, which involve the product of several fractions, we can always simplify the phrase to just one fraction. If a phrase contains several fractions, however, these fractions might be added or subtracted, or divided. We shall see in this section that in all these cases, we may always find another phrase for the same number which involves only one indicated division. We are, thus, able to state one more convention about indicated quotients: No common name for a number shall contain more than one indicated division. Thus, the instruction "simplify" will always include the idea "use the properties of the real numbers to find another name which contains at most one indicated division."

The key to simplifying the sum of two fractions is using the property of one to make the denominators alike.

Example 3. Simplify $\frac{x}{3} + \frac{y}{5}$.

$\frac{x}{3} + \frac{y}{5} = \frac{x}{3}(1) + \frac{y}{5}(1)$, by the property of 1,

$= \frac{x}{3}(\frac{5}{5}) + \frac{y}{5}(\frac{3}{3})$, since $\frac{a}{a} = 1$,

$= \frac{5x}{15} + \frac{3y}{15}$, What Theorem?

$= 5x(\frac{1}{15}) + 3y(\frac{1}{15})$, by the definition of division,

$= (5x + 3y)\frac{1}{15}$, by the distributive property,

$= \frac{5x + 3y}{15}$, by the definition of division.

Once again, you will soon learn to telescope these steps.

Problems

In Problems 1-4 simplify:

1. (a) $\frac{5}{9} + \frac{2}{3}$ (c) $\frac{-5}{9} - \frac{2}{-3}$

   (b) $\frac{5}{9} - \frac{2}{3}$

2. (a) $\frac{4}{a} + \frac{5}{a}$ (c) $\frac{4}{a} + \frac{5}{a^2}$

   (b) $\frac{4}{a} + \frac{5}{2a}$

3. (a) $\frac{x}{4} + \frac{x}{2}$ (c) $\frac{x}{4} - \frac{x}{2}$

   (b) $\frac{x}{4} \cdot \frac{x}{2}$ (d) $\frac{x}{2} - \frac{x}{4}$

4. (a) $\frac{4a}{7} - \frac{a}{35}$ (c) $\frac{4a}{7} - \frac{1}{35}$

   (b) $\frac{4}{7} - \frac{a}{35}$

5. $\frac{x + 8}{10} + \frac{x - 4}{2}$

6. Find the truth set of each of the following open sentences:

Example. $\frac{x}{3} - 2 = \frac{2}{9}x$
Two different procedures are possible.

$$\frac{x}{3} - \frac{2}{9}x = 2$$
$$\frac{3x}{9} - \frac{2x}{9} = 2$$
$$\frac{3x - 2x}{9} = 2$$
$$\frac{x}{9} = 2$$
$$x = 18$$

$$\left(\frac{x}{3} - 2\right)9 = \left(\frac{2}{9}x\right)9 \; *$$
$$3x - 18 = 2x$$
$$x = 18$$

*We multiplied by 9 because we could see that the resulting equation would contain no fractions.

(a) $\frac{1}{4}y + 3 = \frac{1}{2}y$

(b) $\frac{a}{3} + \frac{a}{6} = 1$

(c) $\frac{3 + x}{8} = \frac{15}{24}$

(d) $\frac{7}{9}x = \frac{1}{3}x + 8$

(e) $\frac{3}{4}x = 35 - x$

(f) $3|w| + 8 = \frac{1}{2}|w| + \frac{41}{2}$

(g) $-\frac{3}{7} + |x - 3| < \frac{22}{14}$

7. The sum of two numbers is 240, and one number is $\frac{3}{5}$ times the other. Find the two numbers.

8. The numerator of the fraction $\frac{4}{7}$ is increased by an amount x. The value of the resulting fraction is $\frac{27}{21}$. By what amount was the numerator increased?

9. The sum of two positive integers is 7 and their difference is 3. What are the numbers? What is the sum of the reciprocals of these numbers? What is the difference of the reciprocals?

10. (a) If it takes Joe 7 days to paint his house, what part of the job will he do in one day? How much in d days?

(b) If it takes Bob 8 days to paint Joe's house, what part of the job would he do in one day? In d days?

(c) If Bob and Joe work together what portion of the job would they do in one day? What portion in d days?

(d) Referring to Parts (a), (b), (c), translate the following into an English sentence:

$$\frac{d}{7} + \frac{d}{8} = 1.$$

Solve this open sentence for d. What does d represent?

(e) What portion of the painting will Joe and Bob, working together, do in one day?

11. A ball team on August 1 had won 48 games and lost 52. They had 54 games left on their schedule. Let us suppose that to win the pennant they must finish with a standing of at least .600. How many of their remaining games must they win? What is the highest standing they can get? The lowest?

For simplifying the indicated product of two fractions, a key property was the theorem which stated $\frac{a}{b} \cdot \frac{c}{d} = \frac{ab}{cd}$; for simplifying the indicated sum of two fractions, a key property was the property of 1. When handling the indicated quotient of two fractions, we have several alternative procedures involving these properties. Let us consider an example.

Example 4. Simplify $\dfrac{\frac{5}{3}}{\frac{1}{2}}$.

Method 1. Let us apply the property of 1, where we shall think of 1 as $\frac{6}{6}$. (You will see why we chose $\frac{6}{6}$ as the work proceeds.)

$$\begin{aligned}\frac{\frac{5}{3}}{\frac{1}{2}} &= \frac{\frac{5}{3}}{\frac{1}{2}}\left(\frac{6}{6}\right)\\ &= \frac{\frac{5}{3} \times 6}{\frac{1}{2} \times 6}\\ &= \frac{10}{3},\end{aligned}$$

by our previous work on multiplication.

Method 2. Let us use the property of 1, where we shall think of 1 as $\frac{2}{2}$. We choose 2 because it is the reciprocal of $\frac{1}{2}$.

then $\frac{\frac{5}{3}}{\frac{1}{2}} = \frac{\frac{5}{3}}{\frac{1}{2}}\left(\frac{2}{2}\right)$

$= \frac{\frac{5}{3} \times 2}{\frac{1}{2} \times 2}$

$= \frac{\frac{10}{3}}{1}$ numerator by previous work
denominator by choice of reciprocal of $\frac{1}{2}$

$= \frac{10}{3}$ because $\frac{a}{1} = a$ for any a.

Method 3. Let us apply the definition of division

$\frac{\frac{5}{3}}{\frac{1}{2}} = \frac{5}{3}\left(\frac{1}{\frac{1}{2}}\right)$

$= \frac{5}{3}(2)$ Since $\frac{1}{\frac{1}{a}} = a$

$= \frac{10}{3}$ by previous work on multiplication.

You may apply any one of these methods which appeals to you, provided that (1) you always understand what you are doing, and (2) you receive no instructions to the contrary.

Problems

In each of the following, simplify, using the most appropriate method:

1. $\dfrac{\frac{3}{8}}{\frac{1}{4}}$

2. $\dfrac{\frac{a^2}{2}}{\frac{a}{4}}$

3. $\dfrac{\frac{ax}{y^2}}{\frac{3xy}{a^2}}$

4. $\dfrac{\frac{2}{3} + \frac{1}{6}}{\frac{5}{6}}$

5. $\dfrac{12}{\frac{3}{4} + \frac{1}{2}}$

6. $\dfrac{\frac{7}{8} - \frac{11}{12}}{2}$

7. $\dfrac{\frac{1}{3} + \frac{1}{4}}{\frac{1}{3} - \frac{1}{4}}$

8. $\dfrac{\frac{x + 8}{9}}{\frac{3}{x + 2}}$

9. $\dfrac{\frac{xy + y}{x}}{3 + \frac{3}{x}}$

Summary.

Definition of subtraction: To subtract the real number b from the real number a, add the opposite of b to a.

Theorem. For any real numbers a, b, c, $a = b + c$ if and only if $a - b = c$.

Agreement: $a - b - c = a + (-b) + (-c)$

On the number line.

$a - b$ is the distance from b to a

$b - a$ is the distance from a to b

$|a-b|$ is the distance between a and b.

Definition of division: To divide the real number a by the non-zero real number b, multiply a by the reciprocal of b.

Theorem. For any real numbers a, b, c, where $b \neq 0$, $a = cb$ if and only if $\frac{a}{b} = c$.

Theorem. For any real numbers a, b, c, d, if $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

The simplest name for a number:

(1) Should have no indicated operations which can be performed.

(2) Should in any indicated division have no common factors in the numerator and denominator.

(3) Should have the form $-\frac{a}{b}$ in preference to $\frac{-a}{b}$ or $\frac{a}{-b}$.

(4) Should have at most one indicated division.

FACTORS AND EXPONENTS

Factors and Divisibility. Consider the following quotients: $6 \div 3$; $84 \div 28$; $27 \div 9$; $12 \div 3$; $17 \div 5$. We see that in all but the last of these the division is exact; i.e., there is no remainder. In other words, "3 divides into 6 exactly 2 times", "28 divides into 84 exactly 3 times", and so on. It is a bit clumsy to write "divides into exactly" all the time, so let us use a more compact mathematical term for this. We could say, for instance, that 3 is a "factor" of 12 because $3 \times 4 = 12$; we could also say that 6 is a "factor" of 12 (because $6 \times 2 = 12$). Is 4 also a factor of 12? Is 2?

The number 5, however, is not a factor of 12, because we cannot find another integer such that 5 times that integer equals 12. Of course, 1 and 12 are also factors of 12. Given any positive integer, 1 and the integer itself are factors of that integer; because such factors are always present, they are not very interesting. So we shall call 2 and 3 and 4 and 6 proper factors of 12; these and 1 and 12 are all factors. The number 11, however, does not have any proper factors, because no positive integer other than 1 and 11 is a factor of 11. Now we are ready for a more precise definition of a factor, remembering that a factor of n is one of two integers whose product is n.

> The integer m is a factor of the integer n if $mq = n$, where q is an integer. If the integer q does not equal 1 or n, we say that m is a proper factor of n.

Is 5 a factor of 17?

Does it follow from this definition that if m is a proper factor of n, then m also cannot equal 1 or n?

Since 3 is a factor of 18, then is $\frac{18}{3}$ a factor of 18? Is it true that if m is a factor of n, then $\frac{n}{m}$ is a factor of n? Is the same true for proper factors? How can you tell?

Since 5 is a factor of 15, we say that 5 divides 15. In general, if m and n are positive integers and if m is a factor of n, we say that m divides n, or n is divisible by m. We shall say that 0 is divisible by every integer, but 0 does not divide any number.

Problems

For each of the questions below, if the answer is "Yes", write the number in factored form as in the definition. If the answer is "No", justify in a similar way.

Example. Is 5 a factor of 45? Yes, since $5 \times 9 = 45$
Is 5 a factor of 46? No, since there is no integer q such that $5q = 46$.

1. Is 30 a factor of 510?
2. Is 12 a factor of 204?
3. Is 10 a factor of 100,000?
4. Is 3 a factor of 10,101?
5. Is 12 a factor of 40,404?

If any of the following numbers are factorable (i.e. have proper factors), find such a factor, and find the product which equals the given number and uses this factor.

Example. $69 = 3 \times 23$
67 is not factorable.

6. 85
7. 51
8. 61
9. 122
10. 141

Is there an easy way to tell whether or not 2 is a factor of a number?

Let us now look at the numbers 5 and 10. A number has 10 as a factor if and only if it has both 2 and 5 as factors. Numbers which have 5 as a factor must have decimals which end in 5 or 0, and numbers which have 2 as a factor must be even; hence, if a number is to have both 2 and 5 as a factor its decimal form must end in 0. Can you formulate what we have just said in terms of two sets and members common to both?

Problems

Think about a test to check whether a number is divisible by 4, and also a test for divisibility by 3. The following examples should give you some real hints on the solutions - but don't be disappointed if a simple rule for 3 to be a factor of a number escapes you for a moment, for it is rather tricky.

1. Divisibility by 4: Which of the following numbers have 4 as a factor? 28, 128, 228, 528, 3028; 106, 306, 806, 118, 5618; 72, 572? Do you see the test? How many digits of the number do you have to consider?

2. Divisibility by 3: Which of the following numbers have 3 as a factor? 27, 207, 2007, 702, 270; 16, 106, 601, 61? How about 36, (observe that $3 + 6 = 9$), 306, 351, 315, 513, 5129, (observe that $5 + 1 + 2 + 9 = 17$), 32122? We write

$$\begin{aligned} 2358 &= 2(1000) + 3(100) + 5(10) + 8(1) \\ &= 2(999 + 1) + 3(99 + 1) + 5(9 + 1) + 8(1) \\ &= 2(999) + 3(99) + 5(9) + 2(1) + 3(1) + 5(1) + 8(1) \\ &= \big(2(111) + 3(11) + 5(1)\big)9 + (2 + 3 + 5 + 8) \\ &= (222 + 33 + 5)9 + (2 + 3 + 5 + 8). \end{aligned}$$

The expression $(222 + 33 + 5)9$ is divisible by 3 (Why?); is $2 + 3 + 5 + 8$ divisible by 3? Notice that the sum of the digits is the key to divisibility by 3. Try to formulate this as a rule.

3. If a number is divisible by 9, is it divisible by 3? If a number is divisible by 3, is it divisible by 9?

4. If you know a test for both 2 and 3, what would be a test for 6?

5. Answer the following questions and in each case tell which divisibility tests made your work easier.

   (a) Is 3 a factor of 101,001?

   (b) Is 3 a factor of 37,199?

   (c) Is 6 a factor of 151,821?

   (d) Is 15 a factor of 91,215?

   (e) Is 12 a factor of 187,326,648?

Prime Numbers. We have been talking about factors of positive integers *over* the positive integers, in the sense that when we write

$$mq = n$$

we accept only positive integers for m, n and q. We could, of course, accept negative integers, or any rational numbers, or even any real numbers, as factors. But if you consider these possibilities for a moment, you will see that they do not add much to our understanding. If, for example, you permit negative integers as factors, do you really find anything new? For example, -2, 2, -3, 3 are factors of 6.

You get a different picture if you accept all rational numbers as possible factors of positive integers. The rational number $\frac{2}{7}$, for example, would be a factor of 13, in this extended sense, because $(\frac{2}{7})(\frac{91}{2}) = 13$. Can you think of any non-zero rational number, in fact, which would not be a factor of 13 in this sense? Try $-\frac{17}{3}$, for example. Since $(\frac{-17}{3})(-\frac{39}{17}) = 13$, we find that $-\frac{17}{3}$ is also a rational factor of 13.

You see that if you try factoring positive integers over the rational numbers or over the real numbers, then every number other than zero becomes a factor of every number. Such a kind of factoring, therefore, would not add much to our understanding of the structure of the real number system, and so we shall not consider it further. Usually factoring over the positive integers gives us the most interesting results, and so when we speak of "factoring" a positive integer, we shall always mean over the positive integers.

We have listed below a set of positive integers less than or equal to 100. Cross out the numbers for which 2 is a proper factor and write a 2 below each of these numbers. (For example, ..., $\cancel{8}_2$, 9, $\cancel{10}_2$, ...)

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

What is the first number after 2 which has not been crossed out? It should be 3. Now cross out all numbers which have 3 as a proper factor and write a 3 below each of the numbers. If a number has already been crossed out with a 2 do not cross it out again but skip it. What now is the first number after 3 which has not been crossed out? It should be 5. Cross out numbers which have as proper factor the number 5. Continue the process. After the fifth step your picture should look like this.

1	2	3	$\cancel{4}_2$	5	$\cancel{6}_2$	7	$\cancel{8}_2$	$\cancel{9}_3$	$\cancel{10}_2$
11	$\cancel{12}_2$	13	$\cancel{14}_2$	$\cancel{15}_3$	$\cancel{16}_2$	17	$\cancel{18}_2$	19	$\cancel{20}_2$
$\cancel{21}_3$	$\cancel{22}_2$	23	$\cancel{24}_2$	$\cancel{25}_5$	$\cancel{26}_2$	$\cancel{27}_3$	$\cancel{28}_2$	29	$\cancel{30}_2$
31	$\cancel{32}_2$	$\cancel{33}_3$	$\cancel{34}_2$	$\cancel{35}_5$	$\cancel{36}_2$	37	$\cancel{38}_2$	$\cancel{39}_3$	$\cancel{40}_2$
41	$\cancel{42}_2$	43	$\cancel{44}_2$	$\cancel{45}_3$	$\cancel{46}_2$	47	$\cancel{48}_2$	$\cancel{49}_7$	$\cancel{50}_2$
$\cancel{51}_3$	$\cancel{52}_2$	53	$\cancel{54}_2$	$\cancel{55}_5$	$\cancel{56}_2$	$\cancel{57}_3$	$\cancel{58}_2$	59	$\cancel{60}_2$
61	$\cancel{62}_2$	$\cancel{63}_3$	$\cancel{64}_2$	$\cancel{65}_5$	$\cancel{66}_2$	67	$\cancel{68}_2$	$\cancel{69}_3$	$\cancel{70}_2$
71	$\cancel{72}_2$	73	$\cancel{74}_2$	$\cancel{75}_3$	$\cancel{76}_2$	$\cancel{77}_7$	$\cancel{78}_2$	79	$\cancel{80}_2$
$\cancel{81}_3$	$\cancel{82}_2$	83	$\cancel{84}_2$	$\cancel{85}_5$	$\cancel{86}_2$	$\cancel{87}_3$	$\cancel{88}_2$	89	$\cancel{90}_2$
$\cancel{91}_7$	$\cancel{92}_2$	$\cancel{93}_3$	$\cancel{94}_2$	$\cancel{95}_5$	$\cancel{96}_2$	97	$\cancel{98}_2$	$\cancel{99}_3$	$\cancel{100}_2$

Did the picture change from the fourth step to the fifth step? Why or why not? If you are having difficulty with this question perhaps it would help if you would consider the first number crossed out in each step. How far would the set of numbers have to be extended before the picture after the fifth step would be different from the picture after the fourth step?

In the set of the first 100 positive integers, you have crossed out all the numbers which have proper factors. Thus, the remaining numbers have no proper factors. We call each of these numbers, other than 1, a prime number.

> A prime number is a positive integer greater than 1 which has no proper factors.

Is it possible to find all the prime numbers in the set of positive integers by the method we have just used (called the Sieve of Eratosthenes)? Is it possible to find all the prime numbers less than some given positive integer by this method? What is the next prime number after 97?

Prime Factorization. Let us now return to the Sieve of Eratosthenes and see what else we can learn from it. Consider, for example, the number 63. It is crossed out, and hence, 63 is not a prime. We see from the diagram that 63 was crossed out when we were working with 3. This means, that 3 is the smallest prime factor of 63.

Since 3 is the smallest prime factor of 63, let us divide it out. We obtain 21, and once again we can look in our chart to see if 21 is a prime. We find that it is not, and that in fact 3 is a factor of 21. Divide 21 by 3, and you obtain 7; if you look for 7 on the chart you find that it is not crossed out, so that 7 is a prime and can be factored no further. We have obtained 63 as 3 times 3 times 7; and the significance of this is that these factors of 63 are all primes. In other words, we have succeeded in writing 63 as a product of prime factors: $63 = 3 \times 3 \times 7$.

Let us try the same procedure again with 60. What prime were you considering when you crossed out 60? If you divide 60 by this prime, what do you obtain? Continue the process. What factorization of 60 as a product of primes do you obtain?

Problems

1. Using the Sieve of Eratosthenes, write each of the following numbers as a product of prime factors:

   84, 16, 37, 48, 50, 18, 96, 99, 78, 47, 12.

A positive integer, you see, can be thought of as "made up" of a number of prime factors. Thus, 63 is made up of two 3's and one 7; 60 is made up of two 2's, one 3, and one 5. We shall have many uses for this "prime factorization", as it is called, of a positive integer. But now we face a problem: How do we do the same thing for a number which is not on our diagram? If you are asked for the prime factorization of 144, you might perhaps consider extending the diagram from 100 to 144. But suppose you are asked for 1764?

Let us now see if 1764 is divisible by any integer. We first try 2. (It is convenient to start with the smallest prime factor.) $1764 = 2 \times 882$. So now let us try 882, as if we had the sieve before us. $882 = 2 \times 441$. Now let us work on 441. Since 2 is not a factor of 441, we must test next whether or not 441 is a multiple of 3. If you check 441 for divisibility by 3, you find that 3 divides $(4 + 4 + 1)$, and hence, 3 *is* a factor of 441. We now obtain 441 as 3×147. There is no sense trying the factor 2 on 147, since if 2 were a factor of 147, it would also have been a factor of 441 (why?). But 3 divides 147, and we obtain $147 = 3 \times 49$. 49, in turn, is 7×7, and 7 is a prime number, so that the job is finished. To summarize: We have found that $1764 = 2 \times 2 \times 3 \times 3 \times 7 \times 7$, and this is the prime factorization which we are looking for.

All this writing is rather clumsy; a more compact way to write it is:

$$\begin{array}{r|l}
1764 & 2 \\
882 & 2 \\
441 & 3 \\
147 & 3 \\
49 & 7 \\
7 & 7 \\
1 &
\end{array}$$

Problems

Find the prime factorization of each of the following numbers:

1. 98
2. 432
3. 258
4. 625
5. 180
6. 1024
7. 378
8. 729
9. 825
10. 576
11. 1098

You may have noticed that we have been speaking of "the" prime factorization of a positive integer, as if we were sure that there was only one such factorization. Can you give any convincing reasons why this should be the case? The fact that every positive integer has exactly *one* prime factorization is often called the Fundamental Theorem of Arithmetic.

Adding and Subtracting Fractions. One of the many uses of prime factorization of integers is in addition and subtraction of fractions. It is easy to add or subtract two fractions if their denominators are the same. We have already found it possible to use the property of 1 to change one fraction to an equal fraction having a different denominator. In this way we can change fractions to be added or subtracted into fractions having the same denominator.

To make addition of fractions as easy as possible it is desirable that this common denominator shall be the *least common multiple* of the denominators. We define the least common multiple of two or more given integers as the smallest integer which is divisible by all the given integers.

Consider the problem of simplifying

$$\frac{1}{4} - \frac{3}{10} - \frac{4}{45} + \frac{1}{6}$$

We can readily see that one common multiple of the denominators is their product $4 \times 10 \times 45 \times 6$, or 10,800. This seems very large. Perhaps what we have learned about prime factorization can help us to find the smallest integer which has 4 and 10 and 45 and 6 as factors.

Consider the prime factors of each denominator:

$$\begin{aligned} 4 &= 2 \times 2, \\ 10 &= 2 \times 5, \\ 45 &= 3 \times 3 \times 5, \\ 6 &= 2 \times 3. \end{aligned}$$

Since 4 must be a factor of the common denominator, this denominator must, in its own prime factorization, contain at least two 2's. In order that 10 be a factor of the denominator, the latter's prime factorization must contain a 2 and a 5; we already have a 2 by the previous requirement that 4 be a factor, but we must also include a 5 now. To summarize what we have so far: in order that both 4 and 10 be factors of the denominator, the prime factorization of the denominator must contain at least two 2's and one 5.

Next, we must have 45 as a factor. This means we have to supply two factors of 3 as well as the two 2's and the 5 we already have; we don't need to supply another 5 because we already have one. Finally, to accommodate the factor 6, we need both a 2 and a 3 in the factorization, but we already have each of these.

Conclusion: The denominator will have the prime factorization $2 \times 2 \times 3 \times 3 \times 5$. We need each of these factors, and any more would be unnecessary. What unnecessary factors did the denominator 10,800 contain?

Now that we have found the least common denominator, we can complete the problem of changing each of the fractions in our problem so that it has this denominator. An easy way to do this is to use the factored form of the least common denominator and the factored form of 4. 4 contains two 2's and nothing more, while the common denominator contains two 2's, two 3's and one 5. Thus, to change 4 into the desired denominator, we have to multiply by two 3's and one 5 to supply the missing factors.

$$\frac{1}{4} = \frac{1}{2 \times 2} = \frac{1}{2 \times 2} \times \frac{3 \times 3 \times 5}{3 \times 3 \times 5} = \frac{45}{2 \times 2 \times 3 \times 3 \times 5}.$$

Similarly,

$$\frac{3}{10} = \frac{3}{2 \times 5} = \frac{3}{2 \times 5} \times \frac{2 \times 3 \times 3}{2 \times 3 \times 3} = \frac{54}{2 \times 2 \times 3 \times 3 \times 5}.$$

We can now do the same with $\frac{4}{45}$ and $\frac{1}{6}$. If you have completed the arithmetic correctly, you will have

$$\frac{1}{4} - \frac{3}{10} - \frac{4}{45} + \frac{1}{6} = \frac{45 - 54 - 16 + 30}{2 \times 2 \times 3 \times 3 \times 5}$$

$$= \frac{5}{2 \times 2 \times 3 \times 3 \times 5} = \frac{1}{2 \times 2 \times 3 \times 3} = \frac{1}{36}.$$

What are the advantages of this way of doing the problem? One advantage is the avoidance of big numbers; the denominator is left in factored form until the very end, and you see that we never had to handle any number larger than 54. Another advantage of having the denominator in factored form is that we need only test the resulting numerator for divisibility by the factors of the denominator in order to change the fraction to "lowest terms".

Problems

1. Find the following sums.

(a) $\frac{3}{14} - \frac{4}{35}$

(b) $-\frac{5}{12} - \frac{7}{18}$

(c) $\frac{5}{21} - \frac{3}{91}$

(d) $\frac{3x}{8} + \frac{5x}{36}$

(e) $\frac{1}{6} + \frac{3}{20} - \frac{2}{45}$

(f) $\frac{3k}{10} + \frac{2k}{28} - \frac{k}{56}$

(g) $\frac{x}{3} + \frac{5x}{8} - \frac{11}{70} + \frac{3}{20}$

2. Are the following sentences true?

(a) $\frac{8}{15} < \frac{13}{24}$

(b) $\frac{3}{16} < \frac{11}{64}$

(c) $\frac{14}{63} < \frac{6}{27}$

3. We recall that for any pair of numbers a and b, exactly one of the following must hold: $a > b$, $a = b$, or $a < b$. Put in the symbol for the correct relation for the following pairs of numbers.

(a) $\frac{6}{27}, \frac{5}{28}$

(b) $\frac{2}{3}, \frac{5}{7}$

(c) $\frac{6}{16}, \frac{9}{24}$

(d) $(\frac{1}{2} + \frac{1}{3}), (\frac{11}{12} - \frac{1}{13})$

4. John and his brother Bob each received an allowance of $1.00 per week. One week their father said, "I will pay each of you $1.00 as usual or I will pay you in cents any number less than 100 plus its largest prime proper factor. If you are not too lazy, you have much to gain." What number should they choose?

5. Suppose John's and Bob's father forgot to say proper factor. How much could the boys gain by their father's carelessness?

6. A man is hired to sell suits at the AB Clothing Store. He is given the choice of being paid \$200 plus $\frac{1}{12}$ of his sales or a straight $\frac{1}{3}$ of sales. If he thinks he can sell \$600 worth of suits per month, which is the better choice? Suppose he could sell \$700 worth of suits, which is the better choice? \$1000? What should his sales be so that the offers are equal?

Some Facts About Factors. Suppose that you were looking for two integers with the property that their sum is 22 and their product is 72. One way to find them would be to try all possible ways of factoring 72, and keep looking until you found a pair that met the condition. We are now going to see, however, that factors of numbers have properties which allow us to rule out many possibilities without actually trying them. The prime factorization of 72 is $2 \times 2 \times 2 \times 3 \times 3$. The two factors of 72 which we are seeking must use up the three 2's and two 3's in the prime factorization of 72. Suppose three 2's were all in one factor, and no 2's in the other; that is, either $(2 \times 2 \times 2)(3 \times 3)$ or $(2 \times 2 \times 2 \times 3)(3)$ or $(2 \times 2 \times 2 \times 3 \times 3)(1)$, then one factor would be even, while the other factor would be odd, because it contained no 2's. But the sum of an even and an odd number is odd, and 22 is not odd, that is,

$$(2 \times 2 \times 2) + (3 \times 3) = 17 \quad \text{or}$$
$$(2 \times 2 \times 2 \times 3) + 3 = 27 \quad \text{or}$$
$$(2 \times 2 \times 2 \times 3 \times 3) + 1 = 73.$$

So this split of 72 won't work; we will have to split the three 2's between the two factors, and thus, put two 2's in one factor, and one 2 in the other.

Next, let us look at the 3's. Do we split the two 3's, or do they both go into one of the two factors? We know 22 does not have 3 as a factor; but if we were to split the two 3's in 72 between the two factors of 72, then each would have 3 as a factor, and then the sum would have 3 as a factor. The sum could certainly not be 22.

We have thus, found that the two factors of 72 must be "put together" as follows: one factor contains two 2's while the other factor contains one 2; one factor contains both 3's, while the other contains no 3's. There are only two possibilities; the two 3's go either with the one 2 or with the two 2's, that is, either $(2 \times 2 \times 3 \times 3)(2)$ or $(2 \times 3 \times 3)(2 \times 2)$. But two 2's with two 3's makes 36, which is clearly too big, so that the two 3's go with the one 2 (making 18) and the other two 2's (making 4) form the other factor. Since $(2 \times 3 \times 3) + (2 \times 2) = 22$ and $(2 \times 3 \times 3)(2 \times 2) = 72$, we have our answer.

The kind of reasoning which we have just done depends on two ideas, namely: if 2 is a factor of one of two numbers, and 2 is a factor of their sum, then 2 is a factor of the other number; and if 3 is a factor of one of two numbers and 3 is not a factor of their sum, then 3 is not a factor of the other number.

Let us first prove a similar theorem.

> Theorem. For positive integers b and c, if 2 is a factor of both b and c, then 2 is a factor of $(b + c)$.

Proof. $2q = b$, q an integer, because 2 is a factor of b;
$2p = c$, p an integer, because 2 is a factor of c.

Therefore,

$2q + 2p = b + c$, (Why?)

$2(q + p) = b + c$, distributive property.

Since, $q + p$ is an integer,

2 is a factor of $(b + c)$.

For example, this theorem guarantees that since 2 is a factor of both 6 and 16, it follows that 2 is a factor of (6 + 16). Since 7 is a factor of both 21 and 35, do you think it follows that 7 is a factor of (21 + 35)? If we replace 2 in the theorem by any positive integer a, we can prove the general result.

> Theorem. For positive integers a, b and c, if a is a factor of both b and c, then a is a factor of (b + c).

The proof of this theorem is left to the student as an exercise.

Another useful theorem is

> Theorem. For positive integers a, b and c, if a is a factor of b, and a is not a factor of (b + c), then a is not a factor of c.

Proof. Assume a is a factor of c; then a is a factor of both b and c and, hence, is a factor of (b + c). (Why?) But this contradicts the given fact that a is not a factor of (b + c). Hence, a is not a factor of c.

Since 3 is a factor of 15, and 3 is not a factor of (15 + 8), we are certain that 3 is not a factor of 8.

A third theorem useful in dealing with factors is

> Theorem. For positive integers a, b and c, if a is a factor of b, and a is a factor of (b + c), then a is a factor of c.

Problems

1. The prime factorization of 150 is $2 \times 3 \times 5 \times 5$. Find two numbers whose product is 150 and (a) whose sum is even; (b) whose sum is divisible by 5; (c) whose sum is not divisible by 5.
2. Write the prime factorization of 18. Find two numbers whose product is 18 and whose sum is 9; 11.

3. Write the prime factorization of the first number in each of the following and use it to find two numbers whose product and whose sum are as indicated below.

(a) Product is 288 and sum is 34
(b) " " 972 " " " 247
(c) " " 216 " " " 217
(d) " " 330 " " " 37
(e) " " 500 " " " 62
(f) " " 270 " " " 39

4. The perimeter of a rectangular field is 68 feet and the area is 225 square feet. If the length and width are integers, find them.

5. Show that if y is a positive integer, then y is a factor of $(3y + y^2)$.

6. For what positive integer x is 3 a factor of $6 + 4x$?

7. Refer to the theorems of this section and answer the following:

(a) 3 is a factor of 39 and 39 is a factor of 195. Does it follow that 3 is a factor of 195?

(b) 3 is a factor of 39 and 5 is a factor of 20. Does it follow that $3 + 5$ is a factor of $39 + 20$?

(c) 3 is a factor of 39 and 5 is a factor of 20. Does it follow that $3 \cdot 5$ is a factor of $39 \cdot 20$?

(d) 3 is a factor of 39 and 3 is a factor of 27. Does it follow that 3^2 is a factor of $39 \cdot 27$?

(e) 3 is a factor of 39 and 3 is a factor of 27. Does it follow that 3 is a factor of $39 + 27$?

(f) 3 is a factor of 39. Does it follow that 3^2 is a factor of 39^2?

(g) 3^2 is a factor of 15^2. Does it follow that 3 is a factor of 15?

(h) 3 is a factor of 39 and 5 is a factor of 135. Does it follow that 3 is a factor of 135?

8. Prove the theorems:

(a) For positive integers a, b, c, if a is a factor of b, and b is a factor of c, then a is a factor of c.

(b) For positive integers a, b, c, d, if a is a factor of b, and c is a factor of d, then ac is a factor of bd.

(c) For positive integers a, b, c, if a is a factor of b, and a is a factor of c, then a^2 is a factor of bc.

(d) For positive integers a and b, if a is a factor of b, then a^2 is a factor of b^2.

9. Which theorem in Problem 8 justifies the following:

(a) 25 is a factor of (35)(15).

(b) 6 is a factor of (30)(7).

(c) $(13)^2$ is a factor of (39)(26).

(d) 49 is a factor of (14)(35).

(e) c^2 is a factor of (5c)(9c) if c is a positive integer.

(f) 20ab is a factor of (15)(24)ab if ab is a positive integer.

Introduction to Exponents. We have seen that we can write a positive integer factored into its prime factors, so that, for example,

$$288 = 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3.$$

This notation is somewhat inconvenient because it is so lengthy; it would not be necessary to write the "2" five times if there were some way, more compact than $2 \times 2 \times 2 \times 2 \times 2$, of writing "five 2's multiplied together".

As a first step in this direction, you already know that 3×3 can be written as 3^2. This is pronounced "3 squared"; the "3" indicates that we are going to multiply 3's together, and the "2" that we are going to multiply two of them. How would we write $2 \times 2 \times 2 \times 2 \times 2$ similarly? The number 288 can thus, be written in factored form more compactly as $2^5 \times 3^2$.

In an expression of the form a^n, we need some way of describing the numbers involved. The "a", which indicates which number we are going to use as a factor several times, is called the base; the "n", which indicates how many of the factors "a" we are going to use, is called the exponent. Thus, a^n means a number consisting of n equal factors a; a^n is called a power, or more precisely, the nth power of a. We can write

$$a^n = \underbrace{a \times a \times \ldots \times a.}_{n \text{ factors}}$$

a^2 is read "a squared" or "a square".

a^3 is read "a cubed" or "a cube".

a^n is read "a to the nth power", or just "a to the nth".

Problems

1. Find the prime factorization of each of the following numbers, using exponents wherever appropriate. 64, 80, 128, 81, 49, 32, 27, 56, 243, 512, 625, 768.
2. In describing the number a^n, what kind of number must n be? Must a be?

3. The expression a^b can be thought of as defining a binary operation which, for any two positive integers a and b produces the number a^b. What does it mean to ask if this operation is commutative? Is it? What would it mean to ask whether or not this operation is associative?

Let us extend our notions about exponents. Since we know that the set of real numbers is closed under multiplication, it must be true that $a^3 \cdot a^2$ names a real number. Is there a simpler name? Since a^3 means that a is a factor three times and a^2 means that a is a factor twice, it follows that $a^3 \cdot a^2$ has a as a factor five times. That is,

$$a^3 \cdot a^2 = \overbrace{\underbrace{a \cdot a \cdot a}_{3\text{ factors}} \cdot \underbrace{a \cdot a}_{2\text{ factors.}}}^{5\text{ factors}} = a^5$$

Write simpler names for each of the following: $a^2 \cdot a^3$; $b^3 \cdot b^3$; $3^3 \cdot 3^4$; $(x^2)(x^5)$; $a^4 \cdot a^3 \cdot a^2$; $c^5 \cdot c^8$; $a^2 \cdot b^3$; $2^2 \cdot 3^3$. Suppose we consider the number $a^m \cdot a^n$, where m and n are positive integers.

$$a^m \cdot a^n = \overbrace{\underbrace{a \times a \times a \times \ldots \times a}_{m\text{ factors}} \times \underbrace{a \times a \times a \times \ldots \times a}_{n\text{ factors}}}^{m + n\text{ factors}} = a^{m + n}$$

Does it seem reasonable, therefore, to say that $a^m \cdot a^n$ and $a^{m + n}$ are names for the same number?

Have you noticed that we have been talking about a^2, a^3, a^5, a^7, etc., that is, forms of the type a^n, where n is a positive integer; but we have not mentioned a^1? Certainly, 1 is a positive integer and we shall define $a^1 = a$.

Problems

1. Write simpler names for the following.

Example. $(9x^2)(3x^4) = (3^2x^2)(3x^4)$
$= 3^3x^6$.

(a) $m^3 \cdot m^{11}$

(b) $(x^3)(x^9)$

(c) $(2x)(2x^3)$

(d) $(2x)(2^3x^3)$

(e) $(27a)(3^4a^3)$
(Hint: Replace 27 by its prime factorization.)

(f) $(16a^2)(32a^8)$

(g) $(x^{2a})(x^a)$

(h) $3^4 \cdot 3^2$

(i) $3^4 \cdot 2^3$

(j) $2^5 \cdot 3^2 \cdot 5 \cdot 2^2 \cdot 3^3 \cdot 5^2$

(k) $(3k^2t)(3m^2t)$

In Problems 2-12 tell which sentences are true and which are false and show why in each case.

2. $2^3 + 3^3 = 5^3$
3. $2^3 \cdot 3^3 = 6^3$
4. $2^3 + 3^3 = 6^3$
5. $2^3 \cdot 3^3 = 6^6$
6. $2^3 \cdot 3^3 = 6^9$
7. $2^3 + 2^3 = 2^6$
8. $2^3 + 2^3 = 2^4$
9. $3^3 + 3^3 = 3^4$
10. $3^3 + 3^3 + 3^3 = 3^4$
11. $4^3 + 4^3 + 4^3 = 4^4$
12. $4^3 + 4^3 + 4^3 + 4^3 = 4^4$

13. Write other names for:

(a) $2^3(2^2 + 2)$

(b) $2x^3(2x^2 - 4x^3)$

(c) $(a^2 + 2a^3)(a - a^2)$

Further Properties of Exponents. Now let us examine the fraction $\frac{a^5}{a^3}$, $a \neq 0$. Is there a simpler name for this fraction? From the meaning of a^5 and a^3 it is evident that

$$\frac{a^5}{a^3} = \frac{a \times a \times a \times a \times a}{a \times a \times a}$$

$$= a \times a \times \frac{a \times a \times a}{a \times a \times a}$$

$$= a^2.$$

Write simpler names for: $\frac{x^5}{x^2}$; $\frac{b^2}{b^3}$; $\frac{c^6}{c}$; $\frac{3^7}{3^2}$; $\frac{a^2}{a^2}$; $\frac{m^3}{m^2}$; where none of the variables has the value 0. Can you generalize the results? Suppose we consider $\frac{a^5}{a^3}$ again, but reason in this way:

$$a^5 = a^3 \cdot a^2, \text{ because } a^m \cdot a^n = a^{m+n}.$$

Then

$$\frac{a^5}{a^3} = \frac{1}{a^3} \cdot (a^3 \cdot a^2)$$

$$= (\frac{1}{a^3} \cdot a^3)a^2 \qquad \text{(Why?)}$$

$$= 1 \cdot a^2 \qquad \text{(Why?)}$$

$$= a^2 \qquad \text{(Why?)}$$

That is, if $m > n$,

$$\frac{a^m}{a^n} = \frac{1}{a^n}(a^n \cdot a^{m-n})$$

$$= (\frac{1}{a^n} \cdot a^n)a^{m-n}$$

$$= 1 \cdot a^{m-n}$$

$$= a^{m-n}$$

We specify that $m > n$ because we want $m - n$ to be a positive integer.

If $m = n$,

$$\frac{a^m}{a^n} = \frac{a^m}{a^m} = 1.$$

If $m < n$,

$$\begin{aligned}\frac{a^m}{a^n} &= a^m\left(\frac{1}{a^m \cdot a^{n-m}}\right)\\ &= a^m\left(\frac{1}{a^m}\cdot\frac{1}{a^{n-m}}\right)\\ &= \left(a^m \cdot \frac{1}{a^m}\right)\frac{1}{a^{n-m}}\\ &= 1\cdot\frac{1}{a^{n-m}}\\ &= \frac{1}{a^{n-m}}\end{aligned}$$

To summarize: When $(a \neq 0)$

If $m > n$ then $\frac{a^m}{a^n} = a^{m-n}$. For example, $\frac{6^5}{6^3} = 6^2$.

If $m = n$ then $\frac{a^m}{a^n} = 1$. For example, $\frac{6^5}{6^5} = 1$.

If $m < n$ then $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$. For example, $\frac{6^5}{6^9} = \frac{1}{6^4}$.

Problems

1. Simplify:

(a) $\frac{2^{16}}{2^{12}}$

(b) $\frac{2^{10}}{2^{13}}$

(c) $\frac{2^2\cdot 3^4}{2^5\cdot 3^4}$

(d) $\frac{2^3\cdot 3^5}{2^2\cdot 3^7}$

In Problems 2-5 simplify each expression. (We assume that no variable will have the value 0.)

2. (a) $\frac{2x^6}{2^3x^2}$ (c) $\frac{5b^4}{5b^4}$

(b) $\frac{3^2b^6}{3b^4}$ (d) $\frac{4^2a}{4a^2}$

3. (a) $\frac{a^2b^3c}{a^4b^3c^4}$ (c) $a^2b^3c + a^4b^3c^4$

(b) $(a^2b^3c)(a^4b^3c^4)$

4. (a) $\frac{(5x)(5x)}{5^3x^3}$ (c) $\frac{(5x)(5x)}{5x}$

(b) $\frac{5x(5 + x)}{5^3x^3}$

5. (a) $\frac{288x^2y^3}{48x^6y^6}$ (c) $\frac{63x^2y^3}{28a^6b^6}$

(b) $\frac{54x^2y^3}{153x^6a^6}$

In Problems 6-10 tell which sentences are true and which are false and show why.

6. $\frac{3^2}{2^2} = \frac{3}{2}$

7. $\frac{6^3}{3^3} = 2$

8. $\frac{3^4}{2^4} = (\frac{3}{2})^4$

9. $(\frac{4^3}{3^3})(\frac{3}{4})^3 = 1$

10. $\frac{6^3}{3^3} = 2^3$

11. Why must we be careful to avoid 0 as the value of the variables in Problems 2-5?

Having three properties of exponents for handling division is never as satisfactory as just one which will do the same job. It happens that it is possible to reduce all three to just one, namely:

$$\frac{a^m}{a^n} = a^{m - n},$$

if we drop the condition $m > n$. Let us work some problems in two ways, first, using whichever property of the last section is appropriate and second, using

$$\frac{a^m}{a^n} = a^{m - n}.$$

It is convenient to tabulate the results.

Complete the table.

Compare $\frac{a^7}{a^3} = a^{7 - 3} = a^4$ with $\frac{a^7}{a^3} = a^{7 - 3} = a^4$

Compare $\frac{a^3}{a^3} = 1$ with $\frac{a^3}{a^3} = a^{3 - 3} = a^?$

Compare $\frac{a^3}{a^5} = \frac{1}{a^{5 - 3}} = \frac{1}{a^2}$ with $\frac{a^3}{a^5} = a^{3 - 5} = a^?$

Compare $\frac{a^4}{a^4} = 1$ with $\frac{a^4}{a^4} = a^{4 - 4} = a^?$

Compare $\frac{a^2}{a^3} = \frac{1}{a}$ with $\frac{a^2}{a^3} = a^{2 - 3} = a^?$

We have extended notions of numbers in many instances before; can you now extend your notion of exponents? Examine the above table carefully to answer the following questions:

$$a^0 = ?$$

$$a^{-1} = ?$$

$$a^{-2} = ?$$

Do zero and negative exponents make any sense in our definition of $a^n = a \cdot a \cdot a \cdot \ldots$ to n factors? Of course, it is senseless to think of a as a factor (-3) times. But the previous comparisons suggest a way to write just one property of exponents for division. If we define, for m and n positive integers and $a \neq 0$,

$$a^0 = 1,$$

and

$$a^{-n} = \frac{1}{a^n}, \quad a \neq 0,$$

then

$$\frac{a^m}{a^n} = a^m \cdot \frac{1}{a^n}$$

$$= a^m \cdot a^{-n}$$

$$\frac{a^m}{a^n} = a^{m-n}.$$

Example 1.

$$\frac{7^3}{7^5} = 7^{3-5}$$

$$= 7^{-2}$$

$$= \frac{1}{7^2}$$

Is this the same result you get using the former definition?

Now that we have a meaning for a negative exponent and for a zero exponent, the properties

$$a^m a^n = a^{m+n} \quad \text{and} \quad \frac{a^m}{a^n} = a^{m-n}$$

hold for any integers m and n, whether positive, zero, or negative.

Example 2.

$$\frac{x^{-2}y^{-3}}{x^4y^{-2}} = \left(\frac{x^{-2}}{x^4}\right)\left(\frac{y^{-3}}{y^{-2}}\right), \quad x \neq 0, \quad y \neq 0,$$

$$= x^{-2-4}y^{-3-(-2)}$$

$$= x^{-6}\cdot y^{-1}$$

$$= \frac{1}{x^6}\cdot\frac{1}{y}$$

$$= \frac{1}{x^6y}$$

Example 3.

$$\frac{10^3 \times 10^{-4}}{10^{-5}} = \frac{10^{3+(-4)}}{10^{-5}}$$

$$= \frac{10^{-1}}{10^{-5}}$$

$$= 10^{-1-(-5)}$$

$$= 10^4$$

Problems

1. Simplify each of the following, first by the single property

$$\frac{a^m}{a^n} = a^{m-n}$$

and also in a form using only positive exponents. (Assume none of the variables takes on the value 0.)

Example. $\frac{a^7}{a^9} = a^{-2} = \frac{1}{a^2}$

(a) $\frac{3^5}{3^3}$

(b) $\frac{3^5}{3^8}$

(c) $\frac{b^4}{b^2}$

(d) $\frac{10^5 \times 10^2}{10^8}$

(e) $\frac{10^2 \times 10^4}{10^3}$

(f) $\frac{10^4 \times 10^3}{10^2 \times 10^5}$

(g) $\frac{a^4b^3}{a^7b}$

(h) $\frac{36x^2y^4}{8x^5y}$

(i) $\frac{t^3}{3t^5}$

2. Simplify to a form with only positive exponents. (Assume none of the variables takes on the value 0.)

(a) $\frac{10^4 \times 10^{-2}}{10^2}$

(b) $\frac{10^3 \times 10^2}{10^{-2}}$

(c) $.007 \times 10^4 \times 10^{-4}$

(d) $\frac{12a^4b}{3a^7b^2}$

(e) $\frac{2x^2y^{-2}}{4x^2y^2}$

(f) $\frac{2x^2y^{-2}}{4^2x^2y^2}$

(g) $\frac{3^2 \times 2^{-3}}{2^3 \times 3^{-2}}$

(h) $\frac{10^3 \times 10^{-4} \times 10^0}{10^2 \times 10^{-3}}$

(i) $\frac{2^{-3}x^{-2}y^4}{2^{-2}x^2y^{-1}}$

3. The distance from the earth to the sum in miles is approximately 93,000,000.

(a) How many millions of miles is this?

(b) How many "ten millions" of miles?

(c) Is 9.3×10^7 another name for 93,000,000?

4. In the following, what integral value (or values) of n makes the sentence true?

(a) $10^3 \times 10^3 = 10^n$

(b) $10^{-1} \times 10^{-1} = 10^n$

(c) $10^{-4} \times 10^{-4} = 10^n$

(d) $10^7 \times 10^7 = 10^n$

(e) $10^n \times 10^n = 10^8$

(f) $10^n \times 10^n = 10^{-6}$

(g) $10^n \times 10^n = 10^{18}$

(h) $10^n \times 10^n = 10^{-4}$

5. If n is a positive integer and $a \neq 0$, prove that

$$a^n = \frac{1}{a^{-n}}.$$

What is the meaning of $(ab)^3$? We know ab names a number, and we also know that a number cubed means that the number is a factor three times. Therefore, $(ab)^3$ must mean $(ab)(ab)(ab)$. By the commutative and associative properties of multiplication for real numbers we know that

$$(ab)(ab)(ab) = (aaa)(bbb) = a^3b^3.$$

Thus,

$$(ab)^3 = a^3b^3.$$

Write another name for $(\frac{a}{b})^3$, using similar reasoning. Write another name for $(a^2b^3)^3$ using similar reasoning. In general, we have

$$(ab)^n = a^nb^n.$$

Problems

In Problems 1-8 simplify (assuming no variable has the value 0) and write answers with positive exponents only.

1. (a) $(3a^3)^2$ (c) $(3a^2)^3$

   (b) $3(a^3)^2$ (d) $3a^{(3^2)}$

2. (a) $\frac{5x^2}{15xy^2}$ (c) $\frac{5x^2}{15(xy)^2}$

   (b) $\frac{(5x)^2}{15xy^2}$

3. (a) $\frac{(-3)^2a}{9}$ (c) $\frac{(-3a)^2}{9}$

   (b) $\frac{-3^2a}{9}$ (d) $\frac{(-3a)^3}{9}$

4. $\frac{(2y^2)^3(2y)}{(2y)^3(2y^2)}$

5. $\frac{-7^2z^{15}}{49z^{30}}$

6. $(\frac{28a^3}{45a})(\frac{3}{4})^2$

7. (a) $\frac{x^{2a}}{x^a}$ (c) $(x^{2a})^3$

   (b) $x^{2a}\cdot x^a$

8. $\dfrac{\dfrac{90(ab)^2}{16a^3}}{\dfrac{81ab^3}{108b}}$

9. Is each of the following true? Give reasons for each answer.

(a) $(\frac{2}{3})^2 = \frac{2^2}{3^2}$

(b) $\frac{2}{3} = \frac{2^2}{3^2}$

(c) $(\frac{5a}{7b})^2 = \frac{5^2a^2}{7^2b^2}$

(d) $\frac{5a^2}{7b^2} = \frac{5^2a^2}{7^2b^2}$

(e) 3^3 is a factor of $(3^3 + 3^5)$.

(f) 3^2 is a factor of $(6^2 + 9^2)$.

(g) $(2x + 4y^2)$ is an even number, if x and y are positive integers.

10. (1) Take a number, (2) double it, (3) then square the resulting number. Now start again: (1) take the original number, (2) square it, (3) then double the resulting number.

(a) Is the final result the same in both processes?

(b) Using a variable, show whether or not the two procedures lead to the same result.

11. Simplify the following, that is, change to a form involving one indicated division.

Example.

$$\frac{5}{3x^2} + \frac{11}{6xy} - \frac{4}{9y^2} = \frac{5}{3x^2}\cdot\frac{3\cdot 2y^2}{3\cdot 2y^2} + \frac{11}{6xy}\cdot\frac{3xy}{3xy} - \frac{4}{9y^2}\cdot\frac{2x^2}{2x^2}$$

$$= \frac{30y^2 + 33xy - 8x^2}{3^2\cdot 2x^2y^2}$$

$$= \frac{30y^2 + 33xy - 8x^2}{18x^2y^2}$$

(Notice that $3^2\cdot2\cdot x^2y^2$ is the least common multiple of $3x^2$, $6xy$, and $9y^2$ because $3^2\cdot2\cdot x^2y^2$ is the smallest set of factors which contains $3x^2$, $3\cdot2xy$, and $3\cdot3y^2$.)

(a) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ (b) $\frac{11}{35a^2} + \frac{13}{25ab} - \frac{7}{5b^2}$

12. Prove: If a^2 is odd and if a is an integer, then a is odd. (Hint: Assume a is even and obtain a contradiction.)

13. Prove: If a^2 is even and if a is an integer, then a is even. (Hint: Assume a is odd.)

14. Let a be 2, b be -2, c be 3, d be -3. Then determine the value of:

(a) $-2a^2b^2c^2$ (c) $\frac{-4a^4d}{6b^2a^3}$

(b) $(-2abc)^2$ (d) $\frac{(a+b+c)^2}{a^2+b^2+c^2}$

15. Multiply:

Example.

$$\begin{aligned}(a^2-3)(a^2-2a+1) &= a^2(a^2-2a+1)-3(a^2-2a+1)\\ &= a^4-2a^3+a^2-3a^2+6a-3\\ &= a^4-2a^3-2a^2+6a-3\end{aligned}$$

(a) $(x^2+1)(x^3+x^2+1)$ (c) $(a+b)^3$

(b) $(2x-3y)^2$

POLYNOMIAL AND RATIONAL EXPRESSIONS

Polynomials and Factoring. We have found that there are many advantages to having a numeral in factored form. Consider, for example, the number 288. The common name for this number is actually an abbreviation of "$2(100) + 8(10) + 8$". This is the form on which most of the arithmetic involving the number is based. We also have the factored form "$2^5 \cdot 3^2$". If you want to tell whether or not 288 is a perfect square, which form would you use? What if you wanted to find the simplest form for $\sqrt{288}$? In algebra, the factored form for a positive integer is frequently the most convenient.

Since the prime factored form for integers has turned out to be so useful, it is natural to ask whether we can similarly write algebraic expressions in "factored form", that is, as indicated products of simpler phrases. You have already done problems of this kind. What properties of the real numbers enable us to write, for any real number x,

$$3x^2 + x = x(3x + 1)?$$

We could also write $3x^2 + x$ in the factored form

$$3x^2 + x = (x^2 + 1)\cdot\frac{3x^2 + x}{x^2 + 1}.$$

Why is this latter form not as interesting as the first? One reason is that the factor "$\frac{3x^2 + x}{x^2 + 1}$" is more complicated than the given expression; it involves division, while "$3x^2 + x$" involves only addition and multiplication. We are reminded of our study of positive integers; it was useful to factor positive integers over the positive integers, but not over the rationals or reals.

What type of expression should correspond here to a positive integer? In other words, for what types of expressions will the problem of factoring be interesting? Certainly, phrases such as $3x^2 + x$, x and $3x + 1$ must be included, while phrases such as $\frac{3x^2 + x}{x^2 + 1}$ should be excluded.

Let us look more closely at the form of the phrase

$$3x^2 + x.$$

It involves the integer 3, the variable x, and indicated operations of addition and multiplication. On the other hand, the phrase

$$\frac{3x^2 + x}{x^2 + 1}$$

involves the integers 3 and 1, the variable x, and indicated operations of addition, multiplication and division. As we have seen, the essential difference between these phrases is that the second involves division while the first does not.

Thus, we are led to a general definition of phrases such as "$3x^2 + x$".

> A phrase formed from integers and variables, with no indicated operations other than addition, subtraction, multiplication or taking opposites, is called a polynomial over the integers.

If there is just one variable involved, say x, we have a polynomial in x. Thus, "$3x^2 + x$", "$3x + 1$", "x" are polynomials in x over the integers.

Later we will extend our study to polynomials over the real numbers, but for the time being "polynomial" will mean "polynomial over the integers".

Problems

Which are polynomials over the integers?

1. (a) $3t + 1$ (e) $(x - 2)(x + 3)$

(b) $t + \frac{1}{3}$ (f) $\frac{3}{2}$

(c) $3a^2b$ (g) $rq - \sqrt{2}$

(d) 2 (h) $|x| + 1$

Simplify by performing the indicated multiplications and collecting terms. Is the result always a polynomial over the integers?

2. (a) $2x(x - 2)$ (e) $(u + \frac{1}{2})(u - \frac{1}{2})$

(b) $xy(x - 2y)$ (f) $(x + 2)(x + 2)$

(c) $(t - 2)(t + 3)$ (g) $(3t - 8)(6t + 11)$

(d) $(-3xy^2)(\frac{3}{4}x^2y)z$ (h) $2(y - 1) + y(y - 1)$

3. Will indicated sums and products of polynomials over the integers always be polynomials over the integers?

4. Can an indicated quotient of two polynomials ever be a polynomial? Can such a quotient ever be simplified to be a polynomial? Give an example.

Let us return to the problem of factoring expressions which led us in the first place to consider polynomials. Just as the problem of factoring numbers was most interesting when it was restricted to positive integers, so the problem of factoring expressions is most interesting when it is restricted to polynomials.

Recall the expression "$3x^2 + x$" which we considered at the beginning of this section. This is a polynomial over the integers, and we saw that the distributive property could be used to write it in the factored form

$$3x^2 + x = x(3x + 1).$$

Since "x" and "$3x + 1$" are also polynomials over the integers, we say that we have *factored a polynomial* over the integers.

This suggests the reason for our dislike of the factorization

$$3x^2 + x = (x^2 + 1)\frac{3x^2 + x}{x^2 + 1}.$$

We want the factors of "$3x^2 + x$" to be the *same kind* of phrases as "$3x^2 + x$", namely, *polynomials*. Thus,

> The *problem of factoring* is to write a given polynomial as an indicated product of polynomials.

Just as in the case of positive integers, we also wish to carry the factoring process for polynomials as far as possible, namely, until the factors obtained cannot be factored into "simpler" polynomials.

Factoring can be thought of as the inverse process of what we have called "simplification". For example, given the polynomial

$$x(3x + 5y)(2y - x),$$

we "simplify" it by performing the indicated multiplications and collecting terms, thus, obtaining the polynomial

$$-3x^3 + x^2y - 10xy^2.$$

On the other hand, in order to factor the polynomial

$$-3x^2 + x^2y - 10xy^2,$$

we must somehow reverse the simplification steps so as to obtain

$$x(3x + 5y)(2y - x).$$

By examining carefully the process of simplifying indicated products, we shall work out in this chapter techniques for handling problems of this kind.

Notice that the polynomial obtained in the above simplification is a sum of terms, each of which is also a polynomial. A polynomial which involves at most the taking of opposites and indicated products is called a *monomial*. Hence, each of the terms $-3x^3$, x^2y, $-10xy^2$ is a monomial, and we have written the given polynomial as a sum of monomials. Any polynomial can be written in this way as a sum of monomials.

When a polynomial in *one variable* is written as a sum of monomials, we say its *degree* is the highest power of the variable in any monomial. Thus, for example,

$$3x^2 - 2x + 4$$

is a polynomial of degree two. We also say that "3" is the *coefficient* of x^2, "-2" is the *coefficient* of x, and "4" is the *constant*. A polynomial of degree two is called a *quadratic polynomial*.

In factoring polynomials in one variable, our objective is to obtain polynomial factors of lowest possible degree.

Problems

1. Which of the following polynomials are in factored form?

(a) $(x - 3)(x - 2)$

(b) $(x - 3) + (x - 2)$

(c) $(x - 3)x - 2$

(d) $(x - 3)(x - 2)(x - 1) + (x - 1)$

(e) $(x + y + z)(x - y - z)$

(f) $3z(z + 1) - 2z$

2. Use the distributive property, if possible, to factor as completely as you can each of the following polynomials.

(a) $a^2 + 2ab$

(b) $3t - 6$

(c) $ab + ac$

(d) $3x(xz - yz)$

(e) $ax - ay$

(f) $6p - 12q + 30$

(g) $2(z + 1) - 6zw$

(h) $a^3b^3 + a^2b^4 - a^2b^3$

(i) $3ab + 4bc - 4ac$

(j) $abx - aby$

(k) $(6r^2s)x - (6r^2s)y$

(l) $(u^2 + v^2)x - (u^2 + v^2)y$

(m) $x(4x - y) - y(4x - y)$

(n) $36a^2b^2c^2$

3. What is the degree of each of the following polynomials?

(a) $3x + 2$, $5 - x$, $(3x + 2)(5 - x)$

(b) $x^2 - 4$, $2x + 1$, $(x^2 - 4)(2x + 1)$

(c) $2x^3 - 5x^2 + x$, x^2, $x^2(2x^3 - 5x^2 + x)$

(d) 1, $7x^5 - 6x + 2$, $1\cdot(7x^5 - 6x + 2)$

(e) $x^2 - 3x - 7$, $(x^2 - 3x - 7)^2$

(f) What can you say about the degree of the product of two polynomials if you know the degrees of the polynomials?

Factoring by the Distributive Property. In many of our applications of the distributive property in previous chapters, we changed indicated products into indicated sums and indicated sums into indicated products. The latter is actually factoring and it gives us an important technique for factoring certain polynomials. We saw some simple instances of this in the preceding section. More complicated examples will be considered here.

Example 1. $4t^2 - 6t^4$. This is a polynomial in t. Using the distributive property we write

$$4t^2 - 6t^4 = 2t^2(2 - 3t^2).$$

Example 2. $6a^3b^2 - 3a^2b^3$. This is a polynomial in two variables a and b. The distributive property enables us to write it in various forms, of which some are

$$\begin{array}{rl} \text{(i)} & 3(2a^3b^2 - a^2b^3) \\ \text{(ii)} & ab(6a^2b - 3ab^2) \\ \text{(iii)} & 3a^2(2ab^2 - b^3) \\ \text{(iv)} & 3a^2b^2(2a - b). \end{array}$$

In each case we have factored the polynomial into polynomials. Which factored form is simplest? In Forms (i), (ii), (iii) the distributive property can be used to do more factoring. In the case of (iv), the factoring is complete.

Example 3. $5(z - 2) + (z^2 - 2z)$. This is a polynomial in one variable z over the integers. We first write

$$5(z - 2) + (z^2 - 2z) = 5(z - 2) + z(z - 2).$$

Then by comparing with the form of the distributive property we have

$$\begin{tikzcd}[column sep=small]
 & ac \arrow[dl] \arrow[d] & + & & bc \arrow[dl] \arrow[d] & =(& a \arrow[d] & + & b \arrow[d] &) & c \arrow[d] \\
5 & (\overgroup{z - 2}) & + & z & (\overgroup{z - 2}) & =(& 5 & + & z &) & (\overgroup{z - 2}).
\end{tikzcd}$$

Notice how we identify c with the single number $(z - 2)$. The distributive property has been used twice, first to obtain $z^2 - 2z = z(z - 2)$ and then to write $5(z - 2) + z(z - 2)$ as the indicated product $(5 + z)(z - 2)$.

We found many uses for the factored form of an integer. Similarly, we will find many applications for the factored form of a polynomial.

Example 4. Solve $5(z - 2) + (z^2 - 2z) = 0$.

The result of Example 3 tells us that, for any real number z,

$$5(z - 2) + (z^2 - 2z) = (5 + z)(z - 2).$$

Thus, an equivalent equation is

$$(5 + z)(z - 2) = 0.$$

The truth set of this equation is $\{-5, 2\}$. (Why?) Notice how factorization of polynomials helped us solve the equation.

Problems

Factor each of the following expressions as far as you can using the distributive property. Which cases illustrate factoring polynomials over the integers?

1. $3x(2xz - yz)$
2. $6s^2t - 3stu$
3. $144x^2 - 216s + 180y$. What have you learned to do with integers that will enable you to find the largest common factor here?
4. $\frac{6}{5}u^2v - \frac{9}{5}uv^2 + 3v^2$
5. $-x^3y^2 + 2x^2y^2 + xy^2$
6. $\frac{1}{6}ab + \frac{5}{18}a^2b - \frac{7}{12}ab^2$
7. $s\sqrt{3} + s^2\sqrt{6}$
8. $3ab + 4bc - 5ac$
9. $a(x - 1) + 3(x - 1)$
10. $(x + 3)^2 - 2(x + 3)$
11. $(u + v)x - (u + v)y$
12. $(a - b)a + (a - b)b$
13. $(x + y)(u - v) + (x + y)v$
14. $(r - s)(a + 2) + (s - r)(a + 2)$
15. $3x(x + y) - 5y(x + y) + (x + y)$

The distributive property has enabled us to factor polynomials such as $x^2 + bx$ and $ax + ab$ into

$$x^2 + bx = x(x + b)$$

and

$$ax + ab = a(x + b),$$

respectively. Suppose now that we consider the polynomial

$$x^2 + bx + ax + ab.$$

You see that we can factor the sum of the first two terms, namely, $x^2 + bx$, and the sum of the last two terms, $ax + ab$. Thus,

$$\begin{aligned} x^2 + bx + ax + ab &= (x^2 + bx) + (ax + ab) \\ &= x(x + b) + a(x + b). \end{aligned}$$

We have now succeeded in writing our given sum of four terms as a sum of two terms which have a common factor, $(x + b)$. Applying the distributive property for the third time, we obtain

$$\begin{array}{ccccc} rt & + & st & = & (r + s)\quad t \\ \swarrow\ \downarrow & & \swarrow\ \downarrow & & \downarrow\quad\ \downarrow\quad\ \downarrow \\ x(\overset{\frown}{x + b}) & + & a(\overset{\frown}{x + b}) & = & (x + a)(\overset{\frown}{x + b}), \end{array}$$

$$x^2 + bx + ax + ab = (x + a)(x + b).$$

Factoring such as we have done here, by grouping terms, depends on the arrangement of the terms. For example, consider the arrangement $x^2 + ab + bx + ax$. This can be written as

$$x^2 + ab + bx + ax = (x^2 + ab) + (b + a)x.$$

In this form, however, there is no common factor in the two terms, so it does not lead to a factorization of the given polynomial. (Why?)

Example 5. Factor $x^2 + 4x + 3x + 12$.

$$\begin{aligned} x^2 + 4x + 3x + 12 &= (x^2 + 4x) + (3x + 12) \\ &= x(x + 4) + 3(x + 4). \end{aligned}$$

$$\begin{tikzcd}[column sep=small] a & c & + & b & c & = & (a & + & b) & c \\ x \arrow[from=1-1] & (x + 4) \arrow[from=1-2] & + & 3 \arrow[from=1-4] & (x + 4) \arrow[from=1-5] & = & (x \arrow[from=1-7] & + & 3) \arrow[from=1-9] & (x + 4). \arrow[from=1-10] \end{tikzcd}$$

Thus,

$$x^2 + 4x + 3x + 12 = (x + 3)(x + 4).$$

Again, notice how $(x + 4)$ is treated as a single number when the distributive property is applied the last time.

Example 6. Factor $xz - 8z + x - 8$.

$$\begin{aligned} xz - 8z + x - 8 &= (xz - 8z) + (x - 8) \\ &= (x - 8)z + (x - 8)\cdot 1 \\ &= (x - 8)(z + 1) \end{aligned}$$

Let us try another grouping of terms:

$$\begin{aligned} xz - 8z + x - 8 &= xz + x - 8z - 8 \\ &= (xz + x) - (8z + 8) \\ &= x(z + 1) - 8(z + 1) \\ &= (x - 8)(z + 1) \end{aligned}$$

Example 7. Factor $2st + 6 - 3s - 4t$.

$$\begin{aligned} 2st + 6 - 3s - 4t &= (2st + 6) - (3s + 4t) \\ &= 2(st + 3) - (3s + 4t) \end{aligned}$$

This grouping leads us nowhere. Perhaps another grouping will be better. Notice that $2st$ and $-3s$ have a common factor s, and also the remaining terms 6 and $-4t$ also have a common factor 2. Therefore, we try

$$\begin{aligned} 2st + 6 - 3s - 4t &= 2st - 3s - 4t + 6 \\ &= s(2t - 3) - 2(2t - 3) \\ &= (s - 2)(2t - 3). \end{aligned}$$

We should not conclude that all polynomials of this kind can be factored by the method of Examples 5, 6, 7. Some polynomials which look like these simply cannot be factored regardless of the grouping. For example, try to factor $2st + 6 - 3s - 2t$.

Problems

Factor each of the following polynomials. Consider polynomials over the integers whenever possible.

1. $ax + 2a + 3x + 6$
2. $ux + vx + uy + vy$
3. $2ab + a^2 + 2b + a$
4. $3rs - 3s + 5r - 5$
5. $5x + 3xy - 3y - 5$
6. $3a + 15b - 3a - 15b$
7. $a^2 - ab + ac - bc$
8. $t^2 - 4t + 3t - 12$
9. $p^2 - pq + mp + mq$
10. $2a^2 - 2ab\sqrt{3} - 3ab + 3b^2\sqrt{3}$
11. $2a - 2b + ua - ub + va - vb$. (Try three groups of two terms each.)
12. $x^2 + 4x + 3$. (Note that $4x = 3x + x$.)
13. $a^2 - b^2$. (Note that $a^2 - b^2 = a^2 - ab + ab - b^2$.)

Difference of Squares. Consider, for any two real numbers a and b, the product

$$\begin{aligned}(a + b)(a - b) &= (a + b)a - (a + b)b \\ &= a^2 + ba - ab - b^2 \\ &= a^2 - b^2.\end{aligned}$$

This shows that the product of the sum and difference of any two real numbers is equal to the difference of their squares.

Example 1. Find the product of the sum and difference of 20 and 2.

$$\begin{aligned}(20 + 2)(20 - 2) &= (20)^2 - (2)^2\\ &= 400 - 4\\ &= 396.\end{aligned}$$

Example 2. Find the product of the sum and difference of 2x and 3y.

$$\begin{aligned}(2x + 3y)(2x - 3y) &= (2x)^2 - (3y)^2\\ &= 4x^2 - 9y^2.\end{aligned}$$

Let us turn the above problem around. If we are given the polynomial $a^2 - b^2$, then we know that

$$a^2 - b^2 = (a + b)(a - b).$$

In other words, a difference of squares can be factored into a product of a sum and a difference. Knowing this, we can always factor a polynomial if we can first write it as a difference of squares. Thus, in Example 2, if we are given $4x^2 - 9y^2$, we write

$$\begin{aligned}4x^2 - 9y^2 &= (2x)^2 - (3y)^2\\ &= (2x + 3y)(2x - 3y).\end{aligned}$$

Example 3. Factor $8y^2 - 18$.

Using the distributive property, we have

$$8y^2 - 18 = 2(4y^2 - 9).$$

In this form we recognize one factor to be the difference of squares:

$$\begin{aligned}4y^2 - 9 &= (2y)^2 - (3)^2\\ &= (2y + 3)(2y - 3).\end{aligned}$$

Hence,

$$8y^2 - 18 = 2(2y + 3)(2y - 3).$$

Example 4. Factor $3a^2 - 3ab + a^2 - b^2$.

By grouping, we write

$$\begin{aligned} 3a^2 - 3ab + a^2 - b^2 &= (3a^2 - 3ab) + (a^2 - b^2) \\ &= 3a(a - b) + (a + b)(a - b) \\ &= \big(3a + (a + b)\big)(a - b) \\ &= (3a + a + b)(a - b) \\ &= (4a + b)(a - b) \end{aligned}$$

Example 5. Solve the equation $9x^2 - 4 = 0$.

Since $9x^2 - 4 = (3x + 2)(3x - 2)$ for any real number x, the given equation is equivalent to

$$(3x + 2)(3x - 2) = 0.$$

Moreover, for a real number x, $(3x + 2)(3x - 2) = 0$ if and only if either $3x + 2 = 0$ or $3x - 2 = 0$. Therefore the sentence "$9x^2 - 4 = 0$" is equivalent to the sentence "$3x = 2$ or $3x = -2$", and the truth set of the equation $9x^2 - 4 = 0$ is $\{\frac{3}{2}, -\frac{3}{2}\}$.

Problems

1. Perform the indicated operations.

(a) $(a - 2)(a + 2)$

(b) $(2x - y)(2x + y)$

(c) $(mn + 1)(mn - 1)$

(d) $(3xy - 2z)(3xy + 2z)$

(e) $(a^2 + b^2)(a^2 - b^2)$

(f) $(x - a)(x - a)$

(g) $(2x - y)(x + 2y)$

(h) $(r^2 - s)(r + s^2)$

2. Factor the following polynomials over the integers if possible.

   (a) $1 - n^2$

   (b) $25x^2 - 9$

   (c) $16x^2 - 4y^2$

   (d) $25a^2 - b^2c^2$

   (e) $20s^2 - 5$

   (f) $16x^3 - 4x$

   (g) $49x^4 - 1$

   (h) $x^2 - 4$

   (i) $x^2 - 3$

   (j) $x^2 + 4$

   (k) $3x^2 - 3$

   (l) $(a - 1)^2 - 1$

   (m) $(m + n)^2 - (m - n)^2$

   (n) $(x^2 - y^2) - (x - y)$

3. Solve the equations.

   (a) $x^2 - 9 = 0$

   (b) $9r^2 = 1$

   (c) $75s^2 - 3 = 0$

   (d) $2x^2 = 8$

   (e) $4t^3 - t = 0$

   (f) $x^2 + 4 = 0$

   (g) $y^4 - 16 = 0$

   (h) $(s + 2)^2 - 9 = 0$

4. Factor $20^2 - 1$. Solution:

$$\begin{aligned} 20^2 - 1 &= 20^2 - 1^2 && \text{(Why?)} \\ &= (20 - 1)(20 + 1) && \text{(Why?)} \\ &= 19 \cdot 21. \end{aligned}$$

   Can you see how to reverse these steps? Suppose you are asked to find (19)(21) mentally? Is it easier to find $20^2 - 1$?

   Find mentally:

   (a) (22)(18)

   (b) (37)(43)

   (c) $(26r)(34)$

   (d) $(23x)(17y)$

   (e) (101)(99)

   (f) $(40m)(50n)$

   (g) $\big(36(m + n)\big)\big(44(m - n)\big)$

   (h) (6)(6)(4)(11)

5. (a) Can 899 be a prime number? (Hint: $899 = 30^2 - 1$.)

   (b) Can 1591 be a prime number?

   (c) Can you tell anything about the factors of 391?

   (d) Can you tell anything about the factors of 401?

6. What is $(2-\sqrt{3})(2+\sqrt{3})$? Once again, since we have the sum and difference of the same two numbers, this becomes $(2)^2 - (\sqrt{3})^2 = 4 - 3 = 1$. We can apply this to rationalize the denominator in

$$\frac{1}{2-\sqrt{3}}.$$

By the multiplication property of 1,

$$\frac{1}{2-\sqrt{3}} = \frac{1}{2-\sqrt{3}} \cdot \frac{2+\sqrt{3}}{2+\sqrt{3}}$$
$$= \frac{2+\sqrt{3}}{1} = 2+\sqrt{3}.$$

What does this say about the reciprocal of $2-\sqrt{3}$? Of $2+\sqrt{3}$?

Rationalize the denominator:

(a) $\frac{2}{5+\sqrt{2}}$

(c) $\frac{-6}{2+\sqrt{4}}$

(b) $\frac{3+\sqrt{5}}{3-\sqrt{5}}$

(d) $\frac{3}{\sqrt{6}-\sqrt{5}}$.

7. Factor each of the following:

(a)
$$\begin{aligned} a^3 + b^3 &= a^3 - ab^2 + ab^2 + b^3 \\ &= a(a^2 - b^2) + (a+b)b^2 \\ &= a(a+b)(a-b) + (a+b)b^2 \\ &= (a+b)\big(a(a-b) + b^2\big) \\ &= (a+b)(a^2 - ab + b^2) \end{aligned}$$

(b) $t^3 + 1$

(c) $s^3 + 8$

(d) $27x^3 + 1$

8. Factor each of the following:

(a) $$\begin{aligned} a^3 - b^3 &= a^3 - ab^2 + ab^2 - b^3 \\ &= a(a^2 - b^2) + (a - b)b^2 \\ &= a(a - b)(a + b) + (a - b)b^2 \\ &= (a - b)\big(a(a + b) + b^2\big) \\ &= (a - b)(a^2 + ab + b^2) \end{aligned}$$

(b) $t^3 - 1$

(c) $s^3 - 8$

(d) $8x^3 - 1$

We know that the set of non-rational numbers contains the set of rational numbers. We must determine whether or not numbers like "$2 + \sqrt{3}$" and "$2 - \sqrt{3}$" (any numbers in the form of "$a + b\sqrt{n}$") are closed under the operations of "+" and "x". Are the commutative, associative and distributive properties satisfied?

By now some of you may have the clue to what it is we are doing. Referring back to the section on "Fields and Modulo Systems" in Part 4, you will see that by this time, five of the eight properties of a field have been satisfied. A little more verification of the remaining properties should convince you that the numbers in the form of $a + b\sqrt{n}$ do, in fact, form a field!! (The identity elements for + and × are respectively 0 and 1. Can you write these in the form of $a + b\sqrt{n}$?)

We can now speak more convincingly about the process of "rationalizing the denominator". Since we are working with a field, the operation of dividing one of these numbers by another can be reduced to multiplying the first by the multiplicative inverse of the other. For example,

$$\frac{2 - \sqrt{3}}{5 + \sqrt{3}} = 2 - \sqrt{3} \cdot \frac{1}{5 + \sqrt{3}} .$$

Now since $\frac{1}{5+\sqrt{3}}$ is not in the form of $a + b\sqrt{n}$, our task is to write it in that form so that the multiplication can be performed. We seek another form of $\frac{1}{5+\sqrt{3}}$ where no radicals appear in the denominator. This is accomplished by using the multiplication property of 1 as follows:

$$\frac{1}{5+\sqrt{3}} = \frac{1}{5+\sqrt{3}} \cdot \frac{5-\sqrt{3}}{5-\sqrt{3}}$$

Why do we choose $5 - \sqrt{3}$?

This result is $\frac{5-\sqrt{3}}{22}$ or $\frac{5}{22} - \frac{1}{22}\sqrt{3}$, a number in the form of $a + b\sqrt{n}$. It is now possible to multiply $(2 - \sqrt{3})(\frac{5}{22} - \frac{1}{22}\sqrt{3})$ and obtain a valid result. The whole process, however, is shortened as follows:

$$\frac{2-\sqrt{3}}{5+\sqrt{3}} \cdot \frac{5-\sqrt{3}}{5-\sqrt{3}} = \frac{10 - 7\sqrt{3} + 3}{25 - 3}$$

$$= \frac{13 - 7\sqrt{3}}{22}$$

Perfect Squares. For any real numbers a and b, consider the product

$$\begin{aligned}(a + b)^2 &= (a + b)(a + b)\\ &= (a + b)a + (a + b)b\\ &= a^2 + ba + ab + b^2\\ &= a^2 + 2(ab) + b^2\end{aligned}$$

The polynomial "$a^2 + 2(ab) + b^2$", since it can be written as the product of two identical factors, is called a perfect square. In the same way we can obtain

$$(a - b)^2 = a^2 - 2(ab) + b^2,$$

so that "$a^2 - 2(ab) + b^2$" is also a perfect square.

The problem is to identify a polynomial which is a perfect square and write it in its "squared" form. We have already met perfect squares of the type $25a^4b^2c^6$ $\left(\text{note that } 25a^4b^2c^6 = (5a^2bc^3)^2\right)$. We are interested here in the two types considered above.

Consider the example $(2x + 3y)^2$ in comparison with the general case $(a + b)^2$.

$$\begin{array}{ccccccccc} (a & + & b)^2 & = & a^2 & + & 2(ab) & + & b^2 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow\ \searrow & & \downarrow \\ (2x & + & 3y)^2 & = & (2x)^2 & + & 2(2x\cdot 3y) & + & (3y)^2 \end{array}$$

$$= 4x^2 + 12xy + 9y^2.$$

If we are given $4x^2 + 12xy + 9y^2$ at the outset, the problem is to write it in the form $(2x)^2 + 2(3x\cdot 3y) + (3y)^2$, from which the factored form $(2x + 3y)^2$ is obtained immediately by taking a as $2x$ and b as $3y$ in the general form.

Example 1. Factor $x^2 + 6x + 9$.

$$x^2 + 6x + 9 = x^2 + 2(3x) + 3^2$$
$$= (x + 3)^2$$

How do you tell at a glance whether or not a polynomial such as this is a perfect square?

Example 2. Factor $9s^2 + 12st + 4t^2$.

$$9s^2 + 12st + 4t^2 = (3s)^2 + 2(3s\cdot 2t) + (2t)^2$$
$$= (3s + 2t)^2.$$

Example 3. Factor $4a^2 - 4ab + b^2$.

$$4a^2 - 4ab + b^2 = (2a)^2 - 2(2a\cdot b) + b^2$$
$$= (2a - b)^2.$$

Problems

1. Fill in the missing term so that the result is a perfect square.

   (a) $t^2 - 6t + (\quad)$

   (b) $x^2 + 8x + (\quad)$

   (c) $a^2 + 12a + (\quad)$

   (d) $4s^2 + 4st + (\quad)$

   (e) $(\quad) + 6xy + 9y^2$

   (f) $u^2 - (\quad) + 25$

   (g) $4s^2 + (\quad) + 9$

   (h) $9x^2 + 18x + (\quad)$

2. Which of the following are perfect squares?

   (a) $x^2 + 2xy + y^2$

   (b) $x^2 + 2ax + 9a^2$

   (c) $7^2 + 2(7)(5) + 5^2$

   (d) $7 + 2\sqrt{7}\sqrt{5} + 5$

   (e) $4 - 2\sqrt{15} + \frac{15}{4}$

   (f) $x^2 - \frac{x}{2} + \frac{1}{4}$

   (g) $(x - 1)^2 - 6(x - 1) + 9$

   (h) $(2a + 1)^2 + 10(2a + 1) + 25$

3. Factor each of the following polynomials over the integers, if possible.

   (a) $a^2 - 4a + 4$

   (b) $4x^2 - 4x + 1$

   (c) $x^2 - 4$

   (d) $x^2 + 4$

   (e) $4t^2 + 12t + 9$

   (f) $7x^2 + 14x + 7$

   (g) $y^2 + y + 1$

   (h) $4z^2 - 20z + 25$

   (i) $9s^2 + 6st + 4t^2$

   (j) $9(a - 1)^2 - 1$

4. Write the result of performing the multiplications:

   (a) $(x + 3)^2 =$

   (b) $(x + \sqrt{2})^2 =$

   (c) $(a + b)^2 =$

   (d) $(x - y)^2 =$

   (e) $\big((x - 1) + a\big)\big((x - 1) - a\big) =$

   (f) $(\sqrt{2} + \sqrt{3})^2 =$

   (g) $(100 + 1)^2 =$

Some polynomials can be factored by combining the methods of perfect squares and differences of squares.

Example 4. Factor $x^2 + 6x + 5$.

We know that $x^2 + 6x + 9$ is a perfect square. This suggests that we should write

$$
\begin{aligned}
x^2 + 6x + 5 &= x^2 + 6x + 9 - 9 + 5 \quad \text{(to form a perfect square)} \\
&= (x^2 + 6x + 9) - 4 \\
&= (x + 3)^2 - 2^2 \quad \text{(to form the difference of squares)} \\
&= (x + 3 + 2)(x + 3 - 2) \\
&= (x + 5)(x + 1)
\end{aligned}
$$

The method used above, of adding and subtracting a number so as to obtain a perfect square, is called completing the square.

Example 5. Solve the equation $x^2 - 8x + 18 = 0$.

Completing the square, we obtain

$$x^2 - 8x + 16 - 16 + 18 = 0$$

$$(x - 4)^2 + 2 = 0$$

We know of no method for factoring this polynomial. Does this guarantee that it cannot be factored? In this case, we can find the truth set without writing the polynomial in factored form.

Since $(x - 4)^2$ is non-negative for all x, $(x - 4)^2 + 2$ is never less than 2. Therefore the truth set of the given equation is empty.

Problems

1. Factor the following polynomials over the integers using the method of completing the square.

   (a) $x^2 + 4x + 3$
   (b) $x^2 - 6x + 8$
   (c) $x^2 - 2x - 8$

2. What integer values of p, if any, will make the following polynomials perfect squares?

   (a) $u^2 - 2u + p$
   (b) $x^2 + px + 16$
   (c) $p^2t^2 + 2pt + 1$

3. Solve:

   (a) $y^2 - 10y + 25 = 0$
   (b) $4t^2 - 20t + 25 = 0$
   (c) $9a^2 + 12a + 4 = 0$
   (d) $a^2 = 4a - 4$

Quadratic Polynomials. We have already pointed out that factoring can be regarded as a process inverse to simplification. Thus, it seems plausible that, if someone were to give us a polynomial which was obtained as a result of simplifying a product, we should be able to reverse the process and discover the original factored form. This can be difficult in general but does work in some special cases, as we have already seen in the preceding sections. In this section, we use this approach to factor quadratic polynomials, that is, polynomials in one variable of degree two.

Let us examine the product

$$\begin{aligned}(x + m)(x + n) &= (x + m)x + (x + m)n \\ &= x^2 + mx + xn + mn \\ &= x^2 + (m + n)x + mn\end{aligned}$$

If we are given a quadratic polynomial such as $x^2 + (m + n)x + mn$, where m and n are specific integers, then it is easy to reverse the process:

$$\begin{aligned} x^2 + (m + n)x + mn &= (x^2 + mx) + (nx + mn) \\ &= (x + m)x + (x + m)n \\ &= (x + m)(x + n). \end{aligned}$$

In fact, this is just another example of factoring by the distributive property. However, suppose that m and n are replaced by some common names, say 6 and 4. Then

$$(x + 6)(x + 4) = x^2 + 10x + 24.$$

Now we can see how trouble arises. The variables m and n retain their identity and hold the form of the expression, while 6 and 4 become lost in the simplification. The problem in factoring a polynomial such as $x^2 + 10x + 24$ is to "rediscover" the numbers 6 and 4.

Let us look at this example more closely.

$$\begin{aligned} (x + 6)(x + 4) &= x^2 + (6 + 4)x + (6 \cdot 4) \\ &= x^2 + 10x + 24. \end{aligned}$$

Evidently the problem is to write 24 as a product of two factors whose sum is 10. In this case, since the numbers are simple you can probably list in your mind ways of factoring 24,

$$\begin{gathered} 1 \cdot 24 \\ 2 \cdot 12 \\ 3 \cdot 8 \\ 4 \cdot 6 \end{gathered}$$

and pick the pair of factors whose sum is 10.

Although this method of procedure is easy when the number of factors is small, it becomes tedious when the number of factors is large; on the other hand, the number of cases which need to be considered can frequently be reduced if we use some of our knowledge about integers.

Example 1. Factor the quadratic polynomial $x^2 + 22x + 72$.

We must find two integers whose product is 72 and whose sum is 22. We have $72 = 2^3 3^2$; so the various factors of 72 appear as products of powers of 2 and 3. Since 22 is even, both integers whose sum is 22 must have a factor 2 and, since 22 is not divisible by 3 one of the integers must involve all of the 3's. This reduces the possibilities to

$$2^2 \cdot 3^2 + 2 = 36 + 2$$

or

$$2^2 + 2 \cdot 3^2 = 4 + 18.$$

Since $4 + 18 = 22$, it follows that

$$\begin{aligned} x^2 + 22x + 72 &= x^2 + (4 + 18)x + 4 \cdot 18 \\ &= (x + 4)(x + 18). \end{aligned}$$

Example 2. Factor $x^2 + 5x + 36$.

The prime factorization of 36 is $2^2 \cdot 3^2$. If we examine all possible pairs of factors of 36, we find that the sum can never be as small as 5.

Product	Sum
$1 \cdot 36$	37
$2 \cdot 18$	20
$3 \cdot 12$	15
$4 \cdot 9$	13
$6 \cdot 6$	12

It appears that the smallest sum occurs when the two factors are equal. Since $5 < 12$, we conclude that $x^2 + 5x + 36$ cannot be factored because the coefficient of x is too small.

By similar reasoning determine whether $x^2 - 10x + 36$ is factorable. Is $x^2 + 13x + 49$ factorable? Is $x^2 + 14x + 49$ factorable?

Next try to factor $x^2 + 40x + 36$. In this case 40 is too large for $x^2 + 40x + 36$ to be factorable. (Notice the preceding table of products and sums.)

By similar reasoning determine whether $x^2 - 38x + 36$ is factorable. Is $x^2 + 51x + 49$ factorable? Is $x^2 - 50x + 49$ factorable.

Problems

Factor the quadratic polynomials, if possible, using the above method.

1. (a) $a^2 + 8a + 15$ (c) $a^2 + 2a - 15$
   (b) $a^2 - 8a + 15$ (d) $a^2 - 2a - 15$
2. (a) $t^2 + 12t + 20$ (c) $t^2 + 9t + 20$
   (b) $t^2 + 21t + 20$ (d) $t^2 + 10t + 20$
3. (a) $a^2 + 6a - 55$ (d) $y^2 - 17y - 18$
   (b) $x^2 - 5x + 6$ (e) $z^2 - 2z + 18$
   (c) $u^2 - 10u + 24$
4. (a) $z^6 - 7z^3 - 8$ (c) $a^4 - 13a^2 + 36$
   (b) $b^4 - 11b^2 + 28$ (d) $y^4 - 81$
5. (a) $5a + a^2 - 14$ (c) $108 + a^2 - 21a$
   (b) $10a + 39 + a^2$ (d) $a^2 + 25a - 600$
6. Solve the equations:
   (a) $a^2 - 9a - 36 = 0$ (d) $x^2 + 6x = 0$
   (b) $x^2 = 5x - 6$ (e) $(x - 2)(x + 1) = 4$
   (c) $y^2 - 13y + 36 = 0$ (f) $6x^2 + 6x - 72 = 0$

7. Translate the following into open sentences and find their truth sets:
   (a) The square of a number is 7 greater than 6 times the number. What is the number?
   (b) The length of a rectangle is 5 inches more than its width. Its area is 84 square inches. Find its width.
   (c) The square of a number is 9 less than 10 times the number. What is the number?
8. A rectangular bin is 2 feet deep and its perimeter is 24 feet. If the volume of the bin is 70 cu. ft., what are the length and the width of the bin?
9. Prove that if p and q are integers and if $x^2 + px + q$ is factorable, then $x^2 - px + q$ is also factorable.

In the quadratic polynomials of Examples 1, 2, 3, the coefficient of the second power of the variable was equal to 1. In order to see how to handle other quadratic polynomials let us again consider a product.

$$\begin{aligned}(ax + b)(cx + d) &= (ax + b)cx + ax + b)d \\ &= (ac)x^2 + (ad + bc)x + (bd)\end{aligned}$$

In order to simplify our discussion, let us call a and b the coefficients of the factor $(ax + b)$, c and d the coefficients of $(cx + d)$, and ac, $(ad + bc)$, bd the coefficients of the quadratic polynomial $(ac)x^2 + (ad + bc)x + (bd)$.

Notice how the coefficients of the quadratic polynomial arise. The coefficient of x^2 is the product of the first coefficients of the factors, the constant is the product of the constants of the factors, and the coefficient of x is the product of the "outside" coefficients plus the product of the "inside" coefficients. For example:

$$\underbrace{\overset{2\cdot 3}{(2x + 5)}\overset{5\cdot 2}{(3x + 2)}}_{2\cdot 2 \; + \; 5\cdot 3} = 2\cdot 3x^2 + (2\cdot 2 + 5\cdot 3)x + 5\cdot 2.$$

For simplicity in speaking of these coefficients, we call $2\cdot 3$ the product of the first coefficients, $5\cdot 2$ the product of the last, $2\cdot 2$ the product of the outside and $5\cdot 3$ the product of the inside coefficients.

Thus, the problem of factoring such a quadratic polynomial as "$6x^2 + 19x + 10$" is a problem of finding two factors of 6, and two factors of 10 such that the "sum of the products of the outside and inside factors" is 19. In simple cases, this can be done by checking all possible factorizations of the coefficients. Since the factors of 6 are $1\cdot 6$ or $2\cdot 3$ and the factors of 10 are $1\cdot 10$ or $2\cdot 5$ or $5\cdot 2$ or $10\cdot 1$, we try each possibility.

(1) $$\underbrace{(1x + 1)(6x + 10)}_{1\cdot 10 + 1\cdot 6 = 16}$$

(2) $$\underbrace{(1x + 2)(6x + 5)}_{1\cdot 5 + 2\cdot 6 = 17}$$

(3) $$\underbrace{(1x + 5)(6x + 2)}_{1\cdot 2 + 5\cdot 6 = 32}$$

(4) $$\underbrace{(1x + 10)(6x + 1)}_{1\cdot 1 + 10\cdot 6 = 61}$$

(5) $$(2x + 1)(3x + 10)$$

$$2\cdot 10 + 1\cdot 3 = 23$$

(6) $$(2x + 2)(3x + 5)$$

$$2\cdot 5 + 2\cdot 3 = 16$$

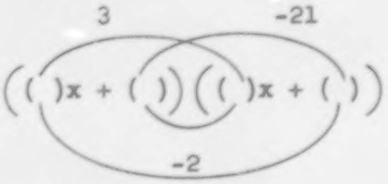
(7) $$(2x + 5)(3x + 2)$$

$$2\cdot 2 + 5\cdot 3 = 19$$

Of course, with a little practice we would have gone directly to the desired factors $(2x + 5)$ and $(3x + 2)$ by eliminating the other cases mentally. We would think: $6 = 2\cdot 3$ and $10 = 2\cdot 5$. Since the middle coefficient 19 is odd, we cannot have an even factor in each of the outside and inside products. This rules out possibilities (1), (3) and (6). Certainly, the factorization $10 = 1\cdot 10$ in the other possibilities will give too large a middle coefficient. This leaves us with possibilities (2) and (7). Hence, we try these two and find that (7) is the desired pair of factors.

Example 3. Factor $3x^2 - 2x - 21$.

We look for coefficients such that



$$\big((\ \ )x + (\ \ )\big)\big((\ \ )x + (\ \ )\big) = 3x^2 - 2x - 21$$

There is one factorization of 3: 3·1, and two factorizations of 21: 21·1 or 3·7. Since 2 is not divisible by 3, we must keep all the 3's in either the outside or inside product. Of the remaining possibilities, which one yields -2 as the sum of the outside and inside products? Hence,

$$3x^2 - 2x - 21 = (x - 3)(3x + 7)$$

Example 4. Factor $25x^2 - 45x - 36$.

We have $25 = 5^2$ and $36 = 2^2 3^2$. We must find a pair of integers whose product is 25, and a pair of integers whose product is -36, such that the sum of the outside and inside products is -45. Since 5 divides 45, there must be a 5 in each of the outside and inside products. Therefore we must have the first coefficients 5 and 5. Since 3 divides 45, there must also be a 3 in each of the outside and inside products. On the other hand, 45 is odd; so the 2's must all occur in one term. We have, thus, reduced the cases to last coefficients: 12, -3; or -12, 3; or 3, -12; or -3, 12. The case which gives the desired factors is: last coefficients -12, 3. Therefore

$$25x^2 - 45x - 36 = \underbrace{\overset{25 \qquad\quad -36}{(5x - 12)(5x + 3)}}_{-45}$$

Problems

Factor, if possible, the polynomials over the integers.

1. (a) $2x^2 + 5x + 3$
   (b) $2x^2 + 7x + 3$
   (c) $2x^2 + 9x + 3$

2. $3a^2 + 4a - 7$
3. $4y^2 + 23y - 6$
4. $3x^2 - 17x - 6$
5. $9x^2 + 12x + 4$
6. $9a^2 + 3a$
7. $10x^2 - 69x - 45$
8. $6 - 23a - 4a^2$
9. $25x^2 - 70xy + 49y^2$

10. Factor:
    (a) $6x^2 - 144x - 150$
    (b) $6x^2 - 11x - 150$
    (c) $6x^2 + 60x + 150$
    (d) $6x^2 - 61x + 150$
    (e) $6x^2 + 25x + 150$
    (f) $6x^2 + 65x + 150$
    (g) $6x^2 - 87x + 150$
    (h) $6x^2 + 63x - 150$

11. Can $2x^2 + ax + b$ be factored if a is even and b is odd? Why?

12. Can $3x^2 + 5x + b$ be factored if 3 is a factor of b? If so, choose a value of b such that $3x^2 + 5x + b$ can be factored.

Translate the following into open sentences and find their truth sets.

13. The sum of two numbers is 15 and the sum of their squares is 137. Find the numbers.

14. The product of two consecutive odd numbers is 15 more than 4 times the smaller number. What are the numbers?

15. Find the dimensions of a rectangle whose perimeter is 28 feet and whose area is 24 square feet.

Polynomials Over the Rational Numbers or the Real Numbers. Most of the preceding work in factoring was concerned with factoring polynomials over the integers. The name itself suggests that we had in mind the possibility of other kinds of polynomials.

> A phrase formed from rational numbers and variables, with no indicated operations other than addition, subtraction, multiplication and taking opposites, is called a polynomial over the rational numbers.

You give a definition for polynomial over the real numbers.

We, thus, have three types of polynomials: polynomials over the integers, over the rational numbers, and over the real numbers. Consider the expression $3x^2 - 4x + 1$. This is a polynomial over the integers. Since every integer is also a rational number, $3x^2 - 4x + 1$ may also be thought of as a polynomial over the rational numbers. Is it possible to regard this as a polynomial over the real numbers? The expression $u^3 - \frac{2}{3}u^2 + u - 1$ is a polynomial over the rational numbers. Is it possible to think of it as a polynomial over the integers? Over the real numbers?

The problem of factoring can now be stated more generally:

> The problem is to write a given polynomial, which we consider to be of a certain type, as an indicated product of polynomials of the same type.

Consider the expression "$x^2 - 2$". This is a polynomial over the integers and, as such, can be factored only in the trivial form

$$x^2 - 2 = (-1)(2 - x^2).$$

This is not especially interesting since the factor $2 - x^2$ is not of lower degree than $x^2 - 2$. In this sense, $x^2 - 2$ is "prime" as a polynomial over the integers. However, $x^2 - 2$ can also be considered as a polynomial over the real numbers, and we have

$$\begin{aligned} x^2 - 2 &= x^2 - (\sqrt{2})^2 \\ &= (x + \sqrt{2})(x - \sqrt{2}), \end{aligned}$$

where $x + \sqrt{2}$ and $x - \sqrt{2}$ are polynomials over the real numbers (but not over the rational numbers or integers). Thus, $x^2 - 2$, considered as a polynomial over the real numbers, admits a non-trivial factoring. This example shows that it makes a difference in factoring which kind of polynomial is being considered.

Consider the expression

$$\text{"}\frac{3}{2}st + \frac{15}{2}st^2 - 27s^2t^3\text{"}.$$

This is a polynomial in two variables over the rational numbers. The distributive property enables us to write it in the form

$$(\tfrac{3}{2})(st + 5st - 18s^2t^3).$$

The factor "$\frac{3}{2}$" may be thought of as a polynomial over the rational numbers, while "$st + 5st - 18s^2t^3$" is a polynomial over the integers. This reduction can always be made:

> A polynomial over the rational numbers can be written as a product of a rational number and a polynomial over the integers.

By this reduction, the problem of factoring polynomials over the rational numbers is reduced to the problem of factoring polynomials over the integers.

The Algebra of Rational Expressions. When we began our discussion of factoring we made a special point of the similarity between factoring polynomials and factoring integers. This similarity can be developed further. The integers are closed under addition, subtraction and multiplication, but not division. The polynomials are closed under (indicated) addition, subtraction and multiplication, but not division. If we extend the system of integers so as also to obtain closure under division (except division by zero) we obtain the system of rational numbers. What is the similar extension for polynomials?

> A *rational expression* is a phrase which involves real numbers and variables with at most the operations of addition, subtraction, multiplication, division and taking opposites.

Is every polynomial a rational expression? Are the rational expressions closed under the indicated operations of addition, subtraction, multiplication and division? Why is $\sqrt{2x - 1}$ not a rational expression?

As examples of rational expressions, we list

(1) $\frac{1}{x} + 1$ (2) $\frac{2x - 3}{4y^2 - 9}$ (3) $\frac{x^3 + 5}{5}$

(4) $\frac{3}{2t} + \frac{5}{s - 1}$ (5) $3a - 2b$ (6) $\frac{z}{z - 1} \cdot \frac{z + 2}{3}$.

Among these rational expressions, (1), (3) and (6) are in one variable. Notice that (2) and (3) are indicated quotients of polynomials, whereas (6) is an indicated product and (1), (4) and (5) are indicated sums of rational expressions. Just as every rational number can be represented as the quotient of two integers, every rational expression can be written as the quotient of two polynomials.

Since rational expressions are phrases, they represent numbers. Therefore, in an expression, such as

$$\frac{3}{2t} + \frac{5}{s - 1},$$

the value of 0 is automatically excluded from the domain of the variable t and the value 1 is excluded from the domain of s. Such a restriction is always understood for any phrase which involves a variable in a denominator.

We are now ready to study some of the "algebra" of rational expressions. This amounts to studying the procedures for simplifying (indicated) sums and products of rational expressions to quotients of polynomials. As was the case for the rational numbers, we might expect some of the work we have done with fractions to apply here. In fact, remember that *for each value of its variables a rational expression is a real number*. Therefore, the same properties hold for operations on rational expressions as hold for operations on real numbers.

For real numbers a, b, c, d we have the properties:

(1) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

(2) $\frac{b}{b} = 1$

(3) $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$.

(where neither b nor d is zero.)

If we symbolize rational expressions with the capital letters A, B, C, D, we can write corresponding properties:

(i) $\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}$

(ii) $\frac{B}{B} = 1$

(iii) $\frac{A}{B} + \frac{C}{B} = \frac{A + C}{B}$.

(where neither B nor D can be written as the zero expression.)

What are the restrictions on the domains of the variables involved in B and D? If A, B are rational expressions and B can be written as the zero expression, then is $\frac{A}{B}$ a rational expression?

We use the above properties applied to rational expressions to simplify the rational expressions. In other words, we want to write an expression as a single indicated quotient of two polynomials which do not have common factors.

Example 1. Simplify $\frac{ax - bx}{x^2} \cdot \frac{a^2 + 2ab + b^2}{a^2 - b^2}$

We use Property (i) with $A = ax - bx$, $B = x^2$, $C = a^2 + 2ab + b^2$ and $D = a^2 - b^2$. Each of these polynomials can be factored:

$A = (a - b)x$ $\qquad$ $C = (a + b)(a + b)$

$B = x \cdot x$ $\qquad$ $D = (a + b)(a - b)$.

Hence,

$$\frac{ax - bx}{x^2} \cdot \frac{a^2 + 2ab + b^2}{a^2 - b^2} = \frac{(a - b)x(a + b)(a + b)}{x^2(a + b)(a - b)}, \quad \text{by (i)}$$

$$= \frac{(a + b)\big(x(a + b)(a - b)\big)}{x\big(x(a + b)(a - b)\big)}$$

$$= \frac{a + b}{x}, \quad \text{by (ii).}$$

The restrictions are

$$x \neq 0, \quad a \neq b, \quad a \neq -b.$$

Example 2.

$$\frac{1 - x^2}{1 + x} \cdot \frac{x - 2}{x^2 - 3x + 2} = \frac{(1 - x)(1 + x)}{1 + x} \cdot \frac{x - 2}{(x - 1)(x - 2)}$$

$$= \frac{(1 - x)(1 + x)(x - 2)}{(1 + x)(x - 1)(x - 2)} \qquad \text{(Why?)}$$

$$= \frac{(1 - x)\big((1 + x)(x - 2)\big)}{(x - 1)\big((1 + x)(x - 2)\big)}$$

$$= \frac{1 - x}{x - 1} \qquad \text{(Why?)}$$

$$= \frac{(-1)(x - 1)}{x - 1} = -1 \qquad \text{(Why?)}$$

What is the restriction on the domain of x?

Example 3.

$$\frac{\dfrac{x^2 + x - 2}{x^2 - 4x + 4}}{\dfrac{x + 2}{x - 2}} = \frac{x^2 + x - 2}{x^2 - 4x + 4} \cdot \frac{x - 2}{x + 2}$$

$$= \frac{(x + 2)(x - 1)(x - 2)}{(x - 2)(x - 2)(x + 2)}$$

$$= \frac{x - 1}{x - 2} \cdot \frac{(x + 2)(x - 2)}{(x + 2)(x - 2)}$$

$$= \frac{x - 1}{x - 2}.$$

Problems

Simplify the following, noting restrictions on the values of the variables:

1. $\frac{3x - 3}{x^2 - 1}$

2. $\frac{x^2 - x^2y}{1 - y}$

3. $\frac{x^2 - 4x - 12}{x^2 - 5x - 6}$

4. $\frac{ab + ab^2}{a - ab^2} \cdot \frac{1 - b}{1 + b}$

5. $\dfrac{\frac{x^2 - 9}{6}}{\frac{x^2 - 3x}{3x + 3}}$

6. $\dfrac{\frac{x^2 + 2x + 1}{x^2 - 1}}{\frac{x + 1}{x - 1}}$

Simplification of Sums of Rational Expressions. In order to use the property

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$$

when adding rational numbers, it is necessary first to write them in a form so that they have a common denominator. In the case of rational numbers, the least (common) denominator which would work was the least common multiple (L.C.M.) of the two given denominators. We have a similar problem in adding rational expressions, and the method is just like that for rational numbers, with factoring of polynomials playing exactly the same role as that of factoring of integers.

Example 1. $\frac{7}{36a^2b} + \frac{5}{24b^3}$

The factored forms of the denominators are

$$36a^2b = 2^2 3^2 a^2 b$$

$$24b^3 = 2^3 3 b^3$$

Choosing each factor the greatest number of times it occurs in either denominator, we find the L.C.M. to be $2^3 \cdot 3^2 a^2 b^3$. Then

$$\frac{7}{36a^2b} + \frac{5}{24b^3} = \frac{7}{2^2 3^2 a^2 b} \cdot \frac{2b^2}{2b^2} + \frac{5}{2^3 3b^3} \cdot \frac{3a^2}{3a^2}$$

$$= \frac{14b^2}{2^3 3^2 a^2 b^3} + \frac{15a^2}{2^3 3^2 a^2 b^3}$$

$$= \frac{14b^2 + 15a^2}{72a^2b^3}$$

Example 2. $\frac{a}{3a - 9} - \frac{2a - 3}{5a - 15}$.

The L.C.M. is $3 \cdot 5(a - 3)$.

If $a \neq 3$, then

$$\frac{a}{3a - 9} - \frac{2a - 3}{5a - 15} = \frac{a}{3(a - 3)} \cdot \frac{5}{5} - \frac{2a - 3}{5(a - 3)} \cdot \frac{3}{3}$$

$$= \frac{5a}{3 \cdot 5(a - 3)} - \frac{6a - 9}{3 \cdot 5(a - 3)}$$

$$= \frac{5a - (6a - 9)}{3 \cdot 5(a - 3)}$$

$$= \frac{5a - 6a + 9}{3 \cdot 5(a - 3)}$$

$$= \frac{-a + 9}{15(a - 3)}$$

Example 3. $(1 - \frac{1}{x + 1})(1 + \frac{1}{x - 1})$

$$1 - \frac{1}{x + 1} = \frac{x + 1}{x + 1} - \frac{1}{x + 1} = \frac{x + 1 - 1}{x + 1} = \frac{x}{x + 1}$$

and

$$1 + \frac{1}{x - 1} = \frac{x - 1}{x - 1} + \frac{1}{x - 1} = \frac{x - 1 + 1}{x - 1} = \frac{x}{x - 1}.$$

Therefore, if $x \neq 1$ and $x \neq -1$, then

$$(1 - \frac{1}{x + 1})(1 + \frac{1}{x - 1}) = \frac{x}{x + 1} \cdot \frac{x}{x - 1} = \frac{x^2}{(x + 1)(x - 1)} = \frac{x^2}{x^2 - 1}.$$

Problems

1. $\frac{3}{x^2} - \frac{2}{5x}$

2. $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

3. $\frac{1}{a^2} - \frac{1}{2a} - 2$

4. $\frac{5}{x - 1} + 1$

5. $\frac{3}{m - 1} + \frac{2}{m - 2}$

6. $\frac{x}{x + 5} - \frac{x}{x - 3}$

7. $\frac{4}{m - n} + \frac{5}{n}$

8. $\frac{x}{x + y} - \frac{y}{x - y}$

9. $\frac{3}{x^2 + 2x} - \frac{5}{3x + 6}$

10. $\frac{4}{a^2 - 4a - 5} + \frac{2}{a^2 + a}$

11. $\dfrac{x - \frac{y^2}{x}}{1 + \frac{y}{x}}$ Hint: Multiply by $\frac{x}{x}$.

12. $\dfrac{\frac{1}{a} + \frac{1}{b}}{\frac{1}{a^2} - \frac{1}{b^2}}$

13. Consider the set of all rational expressions. Do you think this set is closed under each of the four operations of arithmetic?

Division of Polynomials. When you were given a rational number such as $\frac{171}{23}$ in arithmetic you recognized it immediately as an "improper" fraction and hastened to write it in the "proper" form $7\frac{10}{23}$. This form really means $7 + \frac{10}{23}$ and, since $\frac{10}{23} < 1$, *does* have the advantage of telling immediately that $\frac{171}{23}$ lies between the integers 7 and 8. The number 7 is the *integral part* of $\frac{171}{23}$. Since

$$7 + \frac{10}{23} = \frac{7 \cdot 23 + 10}{23} = \frac{171}{23},$$

an equivalent way of looking at this is to write

$$171 = 7 \cdot 23 + 10.$$

Thus, the integer 171 is represented as an integral multiple of 23 plus an integer which is smaller than 23. This is really what we always do when carrying out the process of dividing one integer by another. How do you check your "answer" in division?

We shall now study the similar problem for rational expressions in one variable. Consider the example:

$$\frac{2x^3 - 6x^2 + 5x + 1}{x^2 - 3x} = \frac{2x(x^2 - 3x) + 5x + 1}{x^2 - 3x}$$

$$= 2x + \frac{5x + 1}{x^2 - 3x}.$$

Does this resemble the above example for rational numbers? Notice that we first wrote the numerator in the form

$$2x^3 - 6x^2 + 5x + 1 = 2x(x^2 - 3x) + (5x + 1),$$

that is, as a polynomial multiple of $x^3 - 3x$ plus a polynomial of lower degree than $x^3 - 3x$. Let us lay aside, for the moment, the question of how we did this (later you will learn a systematic way of doing it) and state in general terms just what the problem is.

> Let N and D be two polynomials in one variable. Then to divide N by D means to obtain polynomials Q and R, with R of lower degree than D, such that
>
> $$\frac{N}{D} = Q + \frac{R}{D}.$$

This problem is equivalent to finding polynomials Q and R such that

$$N = QD + R.$$

As in arithmetic, N is the dividend, D the divisor, Q the quotient, and R the remainder. What are N, D, Q, R in the example considered above? It was easy to obtain Q and R in this example since the first two terms of N, $2x^3 - 6x^2$, contain D as a factor.

Our objective is to give a general step-by-step process for finding polynomials Q and R, being given two polynomials N and D. Notice that, since

$$N = QD + R,$$

it follows that

$$R = N - QD.$$

This means that if we can find Q, then R is obtained simply by subtracting QD from N.

Let us consider another example:

$$\frac{2x^2 + x - 5}{x - 3}$$

Here, $N = 2x^2 + x - 5$ and $D = x - 3$. Let us try first to find a polynomial multiple of D which when subtracted from N gives a polynomial of lower degree than N (but not necessarily of lower degree than D). All we need to do is multiply $x - 3$ by a monomial so that the resulting polynomial has the same term of highest degree as N. The highest degree term of N is "$2x^2$". Thus, if we multiply $(x - 3)$ by $2x$, the result has the same term of highest degree.

$$2x(x - 3) = 2x^2 - 6x$$

and

$$(2x^2 + x - 5) - 2x(x - 3) = 7x - 5.$$

That is,

$$(1) \qquad 2x^2 + x - 5 = 2x(x - 3) + (7x - 5).$$

However, the polynomial $7x - 5$ is not of lower degree than $x - 3$. Hence, let us apply the same procedure to $7x - 5$. We multiply $(x - 3)$ by 7 in order to have the same highest degree term as $7x - 5$.

$$(7x - 5) - 7(x - 3) = 16$$

$$(2) \qquad 7x - 5 = 7(x - 3) + 16$$

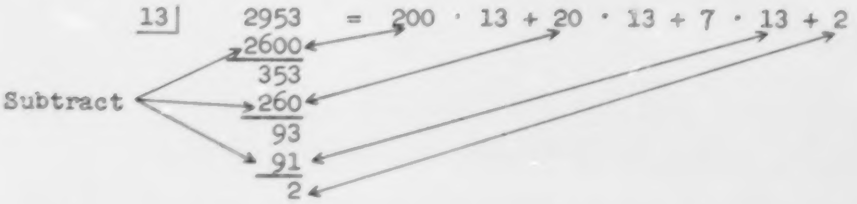
Combining the results (1) and (2), we have

$$\begin{aligned} 2x^2 + x - 5 &= 2x(x - 3) + 7(x - 3) + 16 \\ &= (2x + 7)(x - 3) + 16. \end{aligned}$$

Since 16 has lower degree than $(x - 3)$, the desired polynomials are $Q = 2x + 7$ and $R = 16$. (What _is_ the degree of 16?) Therefore,

$$\frac{2x^2 + x - 5}{x - 3} = (2x + 7) + \frac{16}{x - 3}.$$

In this division process for polynomials we subtract succesively (polynomial) multiples of the divisor, obtaining at each step a polynomial of lower degree. We are finished when the result has lower degree than the divisor. This process has probably recalled to you the familiar long division process for numbers in arithmetic -- remember that long division amounts to successive subtractions of multiples of the divisor from the dividend. For example,



Thus, $$\begin{aligned} 2953 &= 200 \cdot 13 + 20 \cdot 13 + 7 \cdot 13 + 2, \\ &= 13(200 + 20 + 7) + 2 \end{aligned}$$

and

$$\begin{aligned} \frac{2953}{13} &= (200 + 20 + 7) + \frac{2}{13} \\ &= 227 + \frac{2}{13}. \end{aligned}$$

We can use a similar form for arranging our work in dividing polynomials. First, however, we must see how to arrange subtraction "vertically" as is done in arithmetic. For example, the sentence

$$(-5x^4 + 2x^3 - x + 1) - (3x^4 - x^2 + x + 2) = -8x^4 + 2x^3 + x^2 - 2x - 1$$

may be written "vertically" in the form:

$$\begin{array}{lrrrrr} & -5x^4 & +\ 2x^3 & & -\ x & +\ 1 \\ \text{Subtract} \longrightarrow & 3x^4 & & -\ x^2 & +\ x & +\ 2 \\ \hline & -8x^4 & +\ 2x^3 & +\ x^2 & -\ 2x & -\ 1. \end{array}$$

Notice that the terms of the same degree are placed one above the other and if a term (such as, the second degree term in the first polynomial) is missing, then a space is left for it. The difference is then found by subtracting terms of like degree.

Problems

1. Subtract, using the "vertical" form.

(a) $\begin{array}{rrrr} a^3 & -\ 5a^2 & +\ 2a & +\ 1 \\ \hline a^3 & +\ 7a^2 & +\ 9a & -\ 11 \end{array}$ (b) $\begin{array}{rrrrr} -3x^4 & & -\ 5x^2 & -\ 7x & +\ 2 \\ \hline -3x^4 & +\ 2x^3 & -\ 3x^2 & & -\ 6 \end{array}$

2. Use the "vertical form in the following:

(a) Subtract $3a^2 - 6a + 9$ from $3a^2 + 7a - 11$.

(b) From $12x^3 - 11x^2 + 3$ subtract $12x^3 + 6x + 9$.

We can now set up a "vertical" form for dividing polynomials. Let us use the same example as before.

Example 1. $\dfrac{2x^2 + x - 5}{x - 3}$.

$$\begin{array}{lrl} \underline{x - 3|} & 2x^2 + x - 5 & = 2x(x - 3) + 7(x - 3) + 16 \\ & \underline{2x^2 - 6x} & \\ \text{Subtract} & 7x - 5 & \\ & \underline{7x - 21} & \\ & 16 & \end{array}$$

Thus, $2x^2 + x - 5 = (2x + 7)(x - 3) + 16$,

and $\dfrac{2x^2 + x - 5}{x - 3} = 2x + 7 + \dfrac{16}{x - 3}$.

Problems

Perform the indicated divisions, using the form shown in Example 1.

1. $\dfrac{2x^2 - 4x + 3}{x - 2}$
2. $\dfrac{4x^2 - 4x - 15}{2x + 3}$
3. $\dfrac{2x^3 - 2x^2 + 5}{x - 6}$ Hint: Write the dividend $2x^3 - 2x^2 \qquad + 5$ to allow for the missing first degree term.
4. $\dfrac{2x^5 + x^3 - 5x^2 + 2}{x - 1}$
5. $\dfrac{3x^3 - 2x^2 + 14x + 5}{3x + 1}$

Example 1 on page number 277 can be written more compactly as:

$$\begin{array}{rrl}
\text{Dividend} \longrightarrow & \downarrow & \\
\text{Divisor} \longrightarrow \underline{x - 3}\,| & 2x^2 + x - 5 & \underline{|\,2x + 7} \longleftarrow \text{Quotient} \\
& \underline{2x^2 - 6x} & \\
& 7x - 5 & \\
& \underline{7x - 21} & \\
& 16 \longleftarrow & \text{Remainder}
\end{array}$$

Check: $(2x + 7)(x - 3) + 16 = 2x^2 + x - 5.$

Therefore, $\dfrac{2x^2 + x - 5}{x - 3} = 2x + 7 + \dfrac{16}{x - 3}.$

Example 2. Divide $x^3 + 3x^2 - 38x - 10$ by $x - 5$.

$$\begin{array}{rrl}
\underline{x - 5}\,| & x^3 + 3x^2 - 38x - 10 & \underline{|\,x^2 + 8x + 2} \\
& \underline{x^3 - 5x^2 \qquad\qquad\quad} & \\
& 8x^2 - 38x - 10 & \\
& \underline{8x^2 - 40x \qquad} & \\
& 2x - 10 & \\
& \underline{2x - 10} & \\
& 0 &
\end{array}$$

Check: $(x^2 + 8x + 2)(x - 5) + 0 = x^3 + 3x^2 - 38x - 10$

Therefore, $\frac{x^3 + 3x^2 - 38x - 10}{x - 5} = x^2 + 8x + 2$

Problems

1. Divide $x^3 - 3x^2 + 7x - 1$ by $x - 3$.
2. Divide $x^2 - 2x + 15$ by $x - 5$.
3. Divide $5x^3 - 11x + 7$ by $x + 2$.

Perform the indicated division.

4. $\frac{4x^2 - 4x - 15}{2x - 3}$
5. $\frac{6x^3 - x^2 - 5x + 4}{3x - 2}$
6. $\frac{x^4 - 1}{x - 1}$
7. How will the division process tell you when the polynomial D is a factor of N? Show that $x + 3$ is a factor of

$$2x^4 + 2x^3 - 7x^2 + 14x - 3.$$

*In the above examples and problems, we have emphasized the case in which the divisor is a polynomial of degree one. However, the process works equally well with any two polynomials.

Example 3. $\frac{4x^3 - x^2 + 1}{x^4 - 2x^3 + 1}$

There is nothing to do here since N is already of lower degree than D; so $Q = 0$ and $N = R$. In other words, the rational expression is already "proper".

Example 4.

$$\begin{array}{l|l}
\underline{x^3+2} & 5x^4-3x^3+2x^2+12x+1 = 5x(x^3+2)-3(x^3+2)+2x^2+2x+7\\
 & \underline{5x^4 \qquad\qquad\qquad +10x}\\
 & \quad -3x^3+2x^2+2x+1\\
 & \quad \underline{-3x^3 \qquad\qquad\quad -6}\\
 & \qquad\qquad 2x^2+2x+7
\end{array}$$

Thus,

$$5x^4 - 3x^3 + 2x^2 + 12x + 1 = (5x - 3)(x^3 + 2) + 2x^2 + 2x + 7$$

and

$$\frac{5x^4 - 3x^3 + 2x^2 + 12x + 1}{x^3 + 2} = 5x - 3 + \frac{2x^2 + 2x + 7}{x^3 + 2}.$$

Written in the more compact form, this becomes:

$$\begin{array}{l|l|l}
\underline{x^3+2} & 5x^4-3x^3+2x^2+12x+1 & \underline{5x-3}\\
 & \underline{5x^4 \qquad\qquad\qquad +10x} & \\
 & \quad -3x^3+2x^2+2x+1 & \\
 & \quad \underline{-3x^3 \qquad\qquad\quad -6} & \\
 & \qquad\qquad 2x^2+2x+7 &
\end{array}$$

Check: $(5x - 3)(x^3 + 2) + (2x^2 + 2x + 7) = 5x^4 - 3x^3 + 2x^2 + 12x + 1.$

Therefore,

$$\frac{5x^4 - 3x^3 + 2x^2 + 12x + 1}{x^3 + 2} = 5x - 3 + \frac{2x^2 + 2x + 7}{x^3 + 2}.$$

Problems

1. Perform the indicated divisions.

(a) $\dfrac{2x^3 - x^2 + 4x + 5}{x^2 - 3}$ (c) $\dfrac{3x^4 + 4x^3 - 2x^2 - 7}{x^2 - 4}$

(b) $\dfrac{x^3 - 2x^2 + 7x - 1}{x^2 - 2x - 1}$

2. Obtain the second factor in each of the following:

(a) $9x^6 - 25x^4 + 3x + 5 = (3x + 5)(\qquad)$

(b) $x^9 + 1 = (x^3 + 1)(\qquad)$

(c) $2x^4 - 5x^2 - x + 1 = (x^2 - x - 1)(\qquad)$

(d) $4x^8 + 2x^5 - 20x^4 + x^2 - 10x + 25 = (2x^4 + x - 5)(\qquad)$

Summary. We introduced the concept of a polynomial and saw that the problem of factoring expressions is significant only when restricted to factoring polynomials. Although most of our work was with polynomials over the integers, we also considered polynomials over the rational numbers and over the real numbers. Each of these sets of polynomials is closed under addition and multiplication.

In factoring a polynomial of a given type, we insist on factors which are polynomials of the same type. A polynomial which is not factorable as a polynomial over the integers may or may not be factorable when regarded as a polynomial over the real numbers. Factoring a polynomial over the rational numbers can be reduced to factoring a polynomial over the integers.

We found that factoring is a useful tool for solving equations.

The various methods of factoring polynomials are based on the following forms:

Distributive Property:

$$ab + ac = a(b + c).$$

Difference of Squares:

$$a^2 - b^2 = (a + b)(a - b).$$

Perfect Squares:

$$a^2 + 2(ab) + b^2 = (a + b)^2$$

$$a^2 - 2(ab) + b^2 = (a - b)^2$$

Completing the Square:

$$x^2 + px = (x + \frac{p}{2})^2 - \frac{p^2}{4}.$$

Quadratic Polynomials in one variable:

$$x^2 + (m + n)x + (mn) = (x + m)(x + n).$$

$$(ac)x^2 + (ad + bc)x + (bd) = (ax + b)(cx + d).$$

We considered the concept of a rational expression and observed that rational expressions have the same relationship to polynomials as rational numbers have to integers. Problems of simplifying rational expressions are similar to the similar problems for rational numbers. We saw that rational expressions have the usual properties of fractions and that factoring of polynomials plays the same role in the work with rational expressions as factoring of integers plays in the work with rational numbers.

Every rational expression can be written as an indicated quotient of two polynomials which do not have common factors.

We developed a systematic method for division of polynomials in one variable. This is based on the following important property of polynomials:

> For any two polynomials N and D with D different from zero, there exist polynomials Q and R, with R of lower degree than D, such that
>
> $$N = QD + R.$$

The division process gives us a way of calculating Q and R when N and D are given.

TRUTH SETS OF OPEN SENTENCES

Equivalent Open Sentences. Throughout this course we have been solving open sentences, that is, finding their truth sets. At first, we guessed values of the variable which made the sentence true, always checking to verify the truth of the sentence. Later, we learned that certain operations, when applied to the members of a sentence, yielded other sentences with exactly the same truth set as the original sentence. We say that:

> Two sentences are equivalent if they have the same truth set.

Our procedure for solving a sentence then consisted of performing permissible operations on the sentence to yield an equivalent sentence whose truth set is obvious.

What are such permissible operations? Let us recall a problem from our previous work.

Example 1. Solve $3x + 7 = x + 15$.

This sentence is equivalent to

$$(3x + 7) + (-x - 7) = (x + 15) + (-x - 7),$$

that is, to

$$2x = 8.$$

This sentence is equivalent to

$$(2x)(\tfrac{1}{2}) = (8)(\tfrac{1}{2}),$$

that is, to

$$x = 4.$$

Hence, "$3x + 7 = x + 15$" and "$x = 4$" are equivalent sentences, and the desired truth set is $\{4\}$.

Let us examine this example closely. When we say that "$3x + 7 = x + 15$" is equivalent to "$2x = 8$", we mean that every solution of the first sentence is a solution of the second and every solution of the second sentence is a solution of the first. How can we be sure of this? We know that $(-x - 7)$ is a real number for every value of x. Thus, when we add $(-x - 7)$ to both members of the first sentence we obtain another sentence which is true for the same values of x and possibly more. To show equivalence of these sentences we must also verify that every solution of this second sentence is a solution of the first. This we could do by adding $(x + 7)$ to the members of the second sentence to obtain the first sentence, thus, showing that every solution of the second sentence is a solution of the first.

The point is that we did not need to perform this second step of "reversing" the operation. We knew it was possible, because we know that the opposite of $(-x - 7)$ is also a real number for every value of x.

In the same way we know that "$2x = 8$" is equivalent to "$x = 4$", because the operation of multiplying the members of "$2x = 8$" by $\frac{1}{2}$ has an inverse operation, that of multiplying by 2. In fact, every non-zero real number has a reciprocal which is a real number.

Thus, two operations which yield equivalent sentences are:

(1) adding a real number to both members,

(2) multiplying both members by a non-zero real number.

All the sentences we solved, thus far, have been of a type whose members are polynomials. Recall that a polynomial involves no indicated division with variables in the denominator. As a result we have not needed to face the problem of finding simpler sentences by multiplying members of a sentence by expressions involving variables.

Let us now consider other types of sentences, including those whose members are rational expressions.

Example 2. Solve $\frac{x^2}{x^2 + 1} = \frac{1}{2}$.

By multiplying both members by $2(x^2 + 1)$ we can obtain a sentence free of fractions. Will this operation yield an equivalent sentence? Yes, because for every value of x, $2(x^2 + 1)$ is a non-zero real number. Thus, the sentence is equivalent to

$$\frac{x^2}{x^2 + 1} \cdot 2(x^2 + 1) = \frac{1}{2} \cdot 2(x^2 + 1),$$

that is, to

$$2x^2 = x^2 + 1.$$

This sentence is equivalent to

$$x^2 - 1 = 0, \qquad \text{(Why?)}$$

that is, to

$$(x - 1)(x + 1) = 0.$$

Finally, this sentence is equivalent to

$$x - 1 = 0 \text{ or } x + 1 = 0, \qquad \text{(Why?)}$$

and we find the desired truth set to be $\{1, -1\}$.

Problems

1. For each of the following pairs of sentences, determine whether or not the sentences are equivalent. You can prove they are equivalent by beginning with either sentence and applying operations that yield equivalent sentences, until you arrive at the other sentence of the pair. If you think they are not equivalent, try to prove it by finding a number that is in the truth set of one, but not in the truth set of the other.

   (a) $2s = 12$; $s = 6$

   (b) $5s = 3s + 12$; $2s = 12$

   (c) $5y - 4 = 3y + 8$; $y = 6$

   (d) $7s - 5s = 12$; $s = 6$

(e) $2x^2 + 4 = 10$; $x^2 = 4$

(f) $3x + 9 - 2x = 7x - 12$; $\frac{7}{3} = x$

(g) $x^2 = x - 1$; $1 = x - x^2$

(h) $\frac{y - 1}{|y| + 2} = 3$; $y - 1 = 3(|y| + 2)$

(Hint: Is $(|y| + 2)$ a non-zero real number for every value of y?)

(i) $x^2 + 1 = 2x$; $(x - 1)^2 = 0$

(j) $x^2 - 1 = x - 1$; $x + 1 = 1$

(k) $\frac{x^2 + 5}{x^2 + 5} = 0$; $x^2 + 5 = 0$

(l) $\frac{x^2 + 5}{x^2 + 5} = 1$; $x^2 + 5 = 1$

(m) $v^2 + 1 = 0$; $|v + 1| = 0$

2. Change each of the following to a simpler equivalent equation.

(a) $y + 23 = 35$

(b) $\frac{19}{20}x = 19$

(c) $6 - t = 7$

(d) $\frac{1}{7}s = \frac{1}{105}$

(e) $x(x^2 + 1) = 2x^2 + 2$

(Hint: Is $\frac{1}{x^2 + 1}$ a non-zero real number for every value of x?)

(f) $y(|y| + 1) = |y| + 1$

3. Solve (that is, find the truth set of), if possible:

(a) $11t + 21 = 32$

(b) $\frac{4}{3} - \frac{y}{5} = \frac{1}{2}$

(c) $\frac{5}{8}x - 17 = 33$

(d) $6 - s = s + 6$

(e) $s - 6 = 6 - s$

(f) $\frac{y}{3} + \frac{2}{3} = \frac{y}{2} + \frac{3}{2}$

(g) $4x + \frac{3}{2} = x + 6$

(h) $x^4 + x^2 + 1 = x^2$

(i) $y^4+y^3+y^2+y+1 = y^4-y^3+y^2-y+1$

4. Often we can simplify one or both members of a sentence. What kinds of algebraic simplification will guarantee that the simplified form is equivalent to the original? Consider combining terms:

   (a) Are $3x - 2 - 4x + 6 = 0$ and $-x + 4 = 0$ equivalent?
   Consider factoring:

   (b) Are $x^2 - 5x + 6 = 0$ and $(x - 3)(x - 2) = 0$ equivalent?

   (c) Are $\frac{x^2 - 4}{x - 2} = 4$ and $x + 2 = 4$ equivalent?

5. In each of the following pairs of sentences, tell why they are equivalent or why they are not equivalent.

   (a) $2a + 5 - a = 17$; $a + 5 = 17$

   (b) $3x^2 - 6x = 0$; $3x(x - 2) = 0$

   (c) $3x^2 = 6x$; $3x = 6$

   (d) $3x^2 = 6x$; $3x^2 - 6x = 0$

   (e) $6y^2 + 3 - 2y^2 = 5 + y + 2$; $4y^2 + 3 = y + 7$

   (f) $b + 3 = 0$; $0 = b + 3$

   (g) $2 = \frac{y^2 + 2y}{y}$; $2 = y + 2$

   (h) $2(h + 2) + 2(h + 3) = 27$; $4h + 10 = 27$

We have been careful to add only real numbers or multiply only by non-zero real numbers, because we are sure that such operations yield equivalent sentences. Is it possible that other operations may also yield equivalent sentences? Let us look at another example.

Example 3. Solve $x(x - 3) = 2(x - 3)$.

Without any formal operations we can guess that 2 and 3 are solutions of this equation. Are there others? In an attempt to find a simpler equivalent sentence, we might be tempted to multiply both members by $\frac{1}{x - 3}$. Then we obtain the new sentence

$$x(x - 3) \cdot \frac{1}{x - 3} = 2(x - 3) \cdot \frac{1}{x - 3},$$

which in simpler form is

$$x = 2.$$

It is certainly true that 2 is the only solution of this sentence. This means that the operation of multiplying by $\frac{1}{x - 3}$ yielded a new sentence with a smaller truth set. Thus, such an operation will not necessarily give an equivalent sentence. You have probably detected the difficulty: for $x = 3$, $\frac{1}{x - 3}$ is not a number, and the operation of multiplying by $\frac{1}{x - 3}$ is sensible only for values of x other than 3.

Example 3 suggests that we must never add or multiply both members of a sentence by an expression which for some value of the variable is not a number.

Example 4. Solve $\frac{x - 2}{x - 1} = 2 - \frac{1}{x - 1}$.

We first observe that the domain of x cannot include the number 1. (Why?) Thus, we are really solving the sentence

$$\frac{x - 2}{x - 1} = 2 - \frac{1}{x - 1} \text{ and } x \neq 1.$$

It is natural to multiply both members of the equation by $(x - 1)$. Is $(x - 1)$ a real number for every value of x in its domain? Is $(x - 1)$ non-zero? (Remember that $x \neq 1$.) Therefore, we obtain an equivalent sentence by multiplying by $(x - 1)$:

$$\frac{x - 2}{x - 1} \cdot (x - 1) = 2(x - 1) - \frac{1}{x - 1} \cdot (x - 1) \text{ and } x \neq 1,$$

$$x - 2 = 2x - 2 - 1 \text{ and } x \neq 1,$$

$$1 = x \text{ and } x \neq 1.$$

This latter sentence has an empty truth set. Hence, the original sentence has no solutions.

The problem in Example 4 points out that we must be careful to keep a record of the domain of the variable. Thus, we may multiply by an expression which for all values in the domain of the variable is a non-zero real number.

Problems

1. For each of the following phrases decide whether it is
   (i) a real number for every value of the variable,
   (ii) a non-zero real number for every value of the variable.

   (a) $x^2 - 4x + 3$

   (b) $\frac{3 - 4y}{y + 4}$

   (c) $3 + r + \frac{1}{r}$

   (d) $\sqrt{t + 1}$

   (e) $|y + 1|$

   (f) $|y| + 1$

   (g) $\frac{x^2}{x^2 + 1}$

   (h) $\frac{x^2 + 1}{x^2 + 1}$

   (i) $\sqrt{v^2 + 1}$

   (j) -3

   (k) $\frac{x}{x}$

   (l) $\frac{q^2 - 1}{q + 1}$

2. Solve:

   (a) $\frac{y}{y - 2} = 3$

   (b) $\frac{x}{x^2 + 1} = x$

   (c) $\frac{1}{x} + 3 = \frac{2}{x}$

   (d) $\frac{1}{x - 2} + \frac{x - 3}{x - 2} = 2$

   (e) $-\frac{1}{x + 1} + 1 = \frac{x}{x + 1}$

   (f) $x(x^2 + 1) = 2x^2 + 2$

3. Find the dimensions of a rectangle whose perimeter is 30 inches and whose area is 54 square inches.

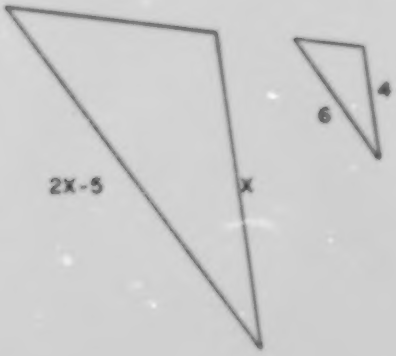
4. Find three successive integers such that the sum of their squares is 61.

5. The sum of two numbers is 8 and the sum of their reciprocals is $\frac{2}{3}$. What are the numbers?

6. In a certain school the ratio of boys to girls was $\frac{7}{6}$. If there were 2600 students in the school, how many girls were there?

7. Show that the equations $4x - \frac{2}{3}y = 6$ and $y = 6x - 9$ are equivalent.

8. The sides of lengths x and $2x - 5$ of the first triangle shown have the same ratio as the sides of lengths 4 and 6 respectively, of the second triangle. How long are the two sides of the first triangle?



Equivalent Inequalities. We have already solved certain inequalities by obtaining simpler equivalent inequalities. Recall that we often used the properties:

For real numbers a, b, c, $a < b$ if and only if $a + c < b + c$,

and

for c positive, $a < b$ if and only if $ac < bc$,
for c negative, $a < b$ if and only if $ac > bc$.

It turns out that the operations we may perform on an inequality to yield an equivalent inequality are somewhat like those for equations. The only difference is that when we multiply both members of an inequality by a non-zero real number, we must be sure that it is positive or that it is negative. For example, $x^2 + 1$ is always positive for every value of x; $-\frac{1}{x^2 + 2}$ is always negative for every value of x; but $x^2 - 1$ is negative for some values, positive for other values, and 0 for others. Hence, we shall not use $x^2 - 1$ as a multiplier.

To summarize, some operations which yield equivalent inequalities are:

(1) adding a real number to both members,

(2) multiplying both members by a positive number, in which case the order of the resulting products is unchanged,

(3) multiplying both members by a negative number, in which case the order of the resulting products is reversed.

Example 1. Solve $\frac{4}{5}y - 6 < \frac{2}{3}y + \frac{5}{6}$.

We may first multiply both members by the positive real number 30 to obtain a sentence free of fractions:

$$24y - 180 < 20y + 25.$$

Now we add the real number $-20y + 180$ to both members:

$$4y < 205.$$

Finally, we multiply by the positive real number $\frac{1}{4}$:

$$y < \frac{205}{4}.$$

What is the truth set of the original inequality? Explain why all these sentences are equivalent.

Example 2. Solve $-\frac{1}{x^2 + 1} > -1$.

Since $-(x^2 + 1)$ is a negative real number for every value of x, we may multiply both members by $-(x^2 + 1)$ to obtain the equivalent sentence

$$1 < x^2 + 1.$$

By adding -1 to both members, we have the equivalent sentence

$$0 < x^2.$$

The truth set of this final sentence is the set of all non-zero real numbers. This is also the truth set of the original inequality.

Problems

1. Solve the following inequalities by changing to simpler equivalent inequalities.

   (a) $x + 12 < 39$

   (b) $\sqrt{2} + 2x > 3\sqrt{2}$

   (c) $8y - 3 > 3y + 7$

   (d) $\frac{t}{3} < 4 + \frac{t}{6} - 2$

   (e) $x^2 + 5 \geq 4$

   (f) $\frac{3}{x^2 + 4} < -2$

   (g) $-\frac{2}{x^2 + 2} \geq -1$

2. Solve the following sentences.

   (a) $1 < 4x + 1 < 2$

   (This is equivalent to "$1 < 4x + 1$ and $4x + 1 < 2$".)

   (b) $4t - 4 < 0$ and $1 - 3t < 0$

   (c) $-1 < 2t < 1$

   (d) $6t + 3 < 0$ or $6t - 3 > 0$

   (e) $|x - 1| < 2$

   (f) $|2t| < 1$

3. Graph the truth sets of the sentences in Problems 2(a), (c) and (3).

4. Solve $3y - x + 7 < 0$ for y; that is, obtain an equivalent sentence with y alone on the left side. What is the truth set for y when $x = 1$? Now solve $3y - x + 7 < 0$ for x. What is the truth set for x if $y = -2$?

5. Write an open sentence expressing that a certain negative number is less than its reciprocal. Solve the sentence.

Equations Involving Factored Expressions. When you solved quadratic equations of the form

$$(x - 3)(x + 2) = 0,$$

you needed this important property of numbers.

> For real numbers a and b, $ab = 0$ if and only if $a = 0$ or $b = 0$.

Restate this property for the particular a and b in the above equation. Interpret the "if and only if" in your own words. It is this property, and the fact that $x - 3$ and $x + 2$ are real numbers for every real number x, that guarantee the equivalence of the sentence "$(x - 3)(x + 2) = 0$" and the sentence "$x - 3 = 0$ or $x + 2 = 0$". Thus, the truth set is $\{3, -2\}$.

How would you extend this property to equations such as $abcd = 0$? State a general property for any number of factors. What is the truth set of

$$(x + 1)(x - 3)(2x + 3)(3x - 2) = 0?$$

Problems

1. Find the truth sets of:
   - (a) $(a + 2)(a - 5) = 0$
   - (b) $(x + 3)(x + 1)(x - 2)(x) = 0$
   - (c) $(3y - 1)(2y + 1)(4y - 3) = 0$
2. Solve:
   - (a) $x^2 - x - 2 = 0$
   - (b) $0 = x^2 - 121$
   - (c) $(x^2 - 1)(x^2 + 5x + 6) = 0$
   - (d) $(x^2 - 5)(x^2 - 24) = 0$

(e) $x^3 = 25x$

(f) $2x^2 - 5x = 3$

(g) $x^3 + x = 2x^2$

3. Find the truth set of the sentence

$$(x - 3)(x - 1)(x + 1) = 0 \text{ and } |x - 2| < 2.$$

We have been careful to avoid adding or multiplying by an expression which for some value of the variable is not a real number. Let us look at another example which illustrates this danger.

Consider this example: Solve $(x - 3)(x^2 - 1) = 4(x^2 - 1)$.

Our first impulse is to multiply both sides by $\frac{1}{x^2 - 1}$. But for some values of x, $\frac{1}{x^2 - 1}$ is not a real number. Which values? Instead, since $4(x^2 - 1)$ __is__ a real number for every x, let us add $-4(x^2 - 1)$ to both members, giving

$$(x - 3)(x^2 - 1) - 4(x^2 - 1) = 0$$

$$(x - 3 - 4)(x^2 - 1) = 0 \qquad \text{(Why?)}$$

$$(x - 7)(x - 1)(x + 1) = 0$$

Each of these sentences is equivalent to every other. What is the resulting truth set? If we had multiplied each side (unthinkingly) by $\frac{1}{x^2 - 1}$, what would be the truth set of the resulting sentence?

This example warns us that "$ac = bc$" and "$a = b$" are __not__ equivalent. Instead, we follow a sequence of __equivalent__ sentences:

$$ac = bc$$

$$ac - bc = 0$$

$$(a - b)c = 0$$

$$a - b = 0 \text{ or } c = 0$$

This tells us that the sentence

$$ac = bc$$

is equivalent to the sentence

$$a - b = 0 \text{ or } c = 0,$$

when a, b, and c are real numbers.

Problems

1. Solve
   (a) $x(2x - 5) = 7x$
   (b) $(3 + x)(x^2 + 1) = 5(3 + x)$
   (c) $(x - 2)(3x + 1) = (x - 2)(x - 5)$
   (d) $3(x^2 - 4) = (4x + 3)(x^2 - 4)$
   (e) $5x - 15 = x^2 - 3x$
2. Multiply both members of the equation "$x^2 = 3$" by $(x - 1)$. Are the new and the original truth sets the same? Is $x - 1$ zero for some value of x?
3. Multiply both members of the equation "$t^2 = 1$" by $(t + 1)$. Compare the new and the original truth sets. Discuss any differences the two multiplications made in the truth sets in Problems 2 and 3.

Fractional Equations. The expression $\frac{1}{x}$ is not a real number when x is 0. Therefore, when we try to solve the equation

$$\frac{1}{x} = 2$$

we are limited to numbers other than 0. In other words, we must solve the sentence

$$\frac{1}{x} = 2 \text{ and } x \neq 0.$$

Knowing that x cannot be 0, we may then multiply by the non-zero number x to obtain

$$\frac{1}{x}\cdot x = 2x \text{ and } x \neq 0,$$
$$1 = 2x \text{ and } x \neq 0.$$

Hence, "$\frac{1}{x} = 2$ and $x \neq 0$" and "$1 = 2x$ and $x \neq 0$" are equivalent sentences. The latter has the truth set $\{\frac{1}{2}\}$. Thus, $\frac{1}{2}$ is the solution.

Another way to handle this same problem is to add -2 to both members of "$\frac{1}{x} = 2$", giving

$$\frac{1}{x} - 2 = 0$$
$$\frac{1 - 2x}{x} = 0 \qquad \text{(Why?)}$$

What are the requirements on a and c for the number $\frac{a}{c}$ to be 0? They are, first, that $c \neq 0$ (Why?), and second, that $a = 0$ (Why?). Thus, the sentence "$\frac{a}{c} = 0$" is equivalent to the sentence "$a = 0$ and $c \neq 0$".

Then "$\frac{1 - 2x}{x} = 0$" is equivalent to what sentence? Your answer should be "$1 - 2x = 0$ and $x \neq 0$", which is the same sentence we had before. Can you find the truth set of "$\frac{x + 1}{x - 2} = 0$" in the same way.

The same two approaches can be used on more complicated fractional equations. Thus, we can solve the equation

$$\frac{1}{x} = \frac{1}{1 - x}$$

either by multiplying both members by a suitable polynomial (What is it?), or by writing it first as $\frac{1}{x} - \frac{1}{1 - x} = 0$ and then simplifying to a single fraction. In either case we must recognize two "illegal values" for x, 0 and 1. The solution is subject to x not taking on those values. Using the second method, we get "$\frac{(1 - x) - x}{x(1 - x)} = 0$" which is equivalent to "$1 - 2x = 0$ and $x \neq 0$ and $x \neq 1$". The solution of this sentence is $\frac{1}{2}$, which is, therefore, the solution of the original sentence.

As a final example, solve

$$\frac{x}{x - 2} = \frac{2}{x - 2}.$$

Since, $x \neq 2$, then upon multiplying both members by $x - 2$ we obtain

$$\frac{x}{x - 2} \cdot (x - 2) = \frac{2}{x - 2} \cdot (x - 2) \text{ and } x \neq 2,$$

$$x = 2 \text{ and } x \neq 2.$$

Hence, the sentence "$\frac{x}{x - 2} = \frac{2}{x - 2}$" is equivalent to the sentence "$x = 2$ and $x \neq 2$". What is the truth set of this sentence?

Problems

Solve the following equations.

1. $\frac{2}{x} - \frac{3}{x} = 10$
2. $\frac{x}{2} - \frac{x}{3} = 10$
3. $x + \frac{1}{x} = 2$
4. $\frac{3}{2y} - \frac{2 + 5y}{y} = \frac{1}{3}$
5. $\frac{1}{y} - \frac{1}{y - 4} = 1$
6. $\left(\frac{x - 1}{x + 1}\right)^2 = 4$
7. $\frac{-2}{x - 2} + \frac{x}{x - 2} = 1$
8. $\left(\frac{x}{x + 1}\right)(x^2 - 1) = 0$
9. $\frac{1 - y}{1 + y} + \frac{1 + y}{1 - y} = 0$
10. $\frac{1 - y}{1 + y} - \frac{1 + y}{1 - y} = 0$

11. (a) Printing press A can do a certain job in 3 hours and press B can do the same job in 2 hours. If both presses work on the job at the same time, in how many hours can they complete it?

    (b) If presses A and C work on the job together and complete it in 2 hours, how long would it take press C to do the job alone?

    (c) Presses A and B begin the job, but at the end of the first hour press B breaks down. If A finishes the job alone, how long does A work after B stops?

Squaring. If $a = b$, then of course $a^2 = b^2$. Why? Do you think it is true, conversely, that if $a^2 = b^2$ then $a = b$? You may see at once that this is not so. Give an example. Hence, "$a^2 = b^2$" and "$a = b$" are not equivalent sentences.

On the other hand, we can alter $a^2 = b^2$ through a chain of equivalent sentences as follows:

$$a^2 = b^2$$

$$a^2 - b^2 = 0$$

$$(a - b)(a + b) = 0$$

$$a - b = 0 \quad \text{or} \quad a + b = 0$$

$$a = b \quad \text{or} \quad a = -b$$

Tell why each of these sentences is equivalent to the next one. Thus, "$a^2 = b^2$" and "$a = b$ or $a = -b$" are equivalent sentences.

If we square both members of the sentence "$x = 3$", we obtain "$x^2 = 9$", which is equivalent to "$x = 3$ or $x = -3$". Thus, squaring the members of a sentence sometimes enlarges the truth set.

Problems

Tell what squaring both members does to the truth sets of the following equations.

1. $x = 2$
2. $x - 1 = 1$
3. $x + 2 = 0$
4. $x - 1 = 2$

In the above problems it is obvious what the original truth set is, and we haven't had to use the new truth set to obtain the old. However, sometimes we square both members of an equation as a simplifying process in situations where we don't already know the truth set. We do know, as in the above problems, that any solution of the original equation is a solution of the equation obtained by squaring. But we also know that the new truth set may be larger than the old. Therefore, *each solution of the new equation must be checked in the original equation* in order to eliminate any possible extra solutions that may have crept in during the squaring.

Example 1. Solve $\sqrt{x + 3} = 1$.

If $\sqrt{x + 3} = 1$ is true for some x, then

$(\sqrt{x + 3})^2 = (1)^2$ is true for the same x;

$$x + 3 = 1,$$

$$x = -2.$$

If $x = -2$, then $\sqrt{x + 3} = \sqrt{-2 + 3} = \sqrt{1} = 1$.

Hence, -2 is the solution.

Example 2. Solve $\sqrt{x} + x = 2$.

Our objective is to square both members and obtain an equation free of radicals. Let us try it.

$$(\sqrt{x} + x)^2 = 2^2$$

$$(\sqrt{x})^2 + 2(\sqrt{x})(x) + x^2 = 2^2$$

$$x + 2x\sqrt{x} + x^2 = 4.$$

Apparently, we have arrived at a more complicated sentence which still contains a radical. Instead, let us write the sentence in the equivalent form

$$\sqrt{x} = 2 - x$$

before squaring its members. Then we obtain

$$(\sqrt{x})^2 = (2 - x)^2$$

$$x = 4 - 4x + x^2$$

$$0 = 4 - 5x + x^2$$

$$0 = (x - 4)(x - 1)$$

$$x = 4 \text{ or } x = 1.$$

In other words, if there are solutions of the sentence they must be in the set $\{1, 4\}$. Checking each of these possibilities, we find that 4 does not make the original sentence true, while 1 does. The solution is therefore, 1.

Example 3. Solve $|x| - x = 1$.

Again we can obtain a simpler sentence by squaring. Here we use a fact about absolute values which you should prove in Problem 9: $|x|^2 = x^2$ for every real number x. Then we have the sequence of sentences

$$\begin{aligned} |x| &= x + 1 \\ (|x|)^2 &= (x + 1)^2 \\ x^2 &= x^2 + 2x + 1 \\ 2x + 1 &= 0 \\ x &= -\frac{1}{2} \end{aligned}$$

Checking back, we find that $-\frac{1}{2}$ does make the original equation true and is, therefore, its solution.

Problems

Solve the following equations by squaring.

1. $\sqrt{2x} = 1 + x$
2. $\sqrt{2x + 1} = x + 1$
3. $\sqrt{x + 1} - 1 = x$
4. $\sqrt{4x} - x + 3 = 0$
5. $3\sqrt{x + 13} = x + 9$
6. $|2x| = x + 1$
7. $2x = |x| + 1$
8. $|x - 2| = 3$
9. Prove: For every real number x,

$$|x|^2 = x^2.$$

10. The distance between x and 3 on the number line is 2 more than x. Solve for x.

11. The time t in seconds it takes a body to fall from rest a distance of s feet is given by the formula $t = \sqrt{\frac{2s}{g}}$. Find s if $t = 6.25$ seconds and $g = 32$.

Graphs of Open Sentences With Two Variables. If we assign the values 0 and -2 to the variables of the open sentence

$$3y - 2x + 6 = 0,$$

is it then a true sentence? To which variable did you assign 0? To which -2? Were there two different ways to make these assignments?

To avoid the kind of confusion you met in the preceding paragraph, let us agree that whenever we write an open sentence with two variables, we must indicate which of the variables is to be considered first. When the variables are x and y, as in the above example, x will always be considered first.

With this agreement we are ready to examine the connection between an open sentence with two ordered variables and an ordered pair of real numbers. Among the set of all ordered pairs of real numbers, each pair has a first number which we associate with the first variable and a second number which we associate with the second variable. In this way, an open sentence with two ordered variables acts as a sorter--it sorts the set of all ordered pairs of real numbers into two subsets:

(1) the set of ordered pairs which make the sentence true, and (2) the set of ordered pairs which make the sentence false. As before, we call this first set the truth set of the sentence.

Now we can answer the question in the first paragraph if we specify the ordered pair (0, -2). Does the ordered pair (0, -2) belong to the set of ordered pairs for which

$$3y - 2x + 6 = 0$$

is a true sentence? Does the ordered pair (-2, 0) belong to the truth set?

An ordered pair belonging to the truth set of a sentence with two variables is called a solution of the sentence, and this ordered pair is said to satisfy the sentence. If r is taken as the first variable, what are some solutions of

$$s = r + 1?$$

Does the ordered pair (-2, -3) satisfy this sentence? Is (-3, -2) a solution?

If u is taken as the first variable, what are some ordered pairs in the truth set of

$$v = 2u^2?$$

Is (-1, 2) a solution of this sentence? Does (2, -1) satisfy this sentence?

Throughout this chapter we shall use only x and y as variables, in order to focus attention on properties of sentences with two ordered variables. But many times in the future you will see other variables used, and then you must always decide which variable is used first.

One other point needs to be stressed. The sentence "$y = 4$" can be considered as a sentence in one variable y, or it can be considered as a sentence with two ordered variables x and y. When we say that "$y = 4$" is a sentence with two variables, we mean that "$y = 4$" is an abbreviation for

$$(0)x + (1)y = 4.$$

What are some solutions of this sentence? What is true of every ordered pair satisfying this sentence? If "$x = -2$" is a sentence with two variables, then "$x = -2$" is an abbreviation for

$$(1)x + (0)y = -2.$$

What is true of every ordered pair satisfying this sentence?

Remember that every point of the plane has an associated pair of numbers called its coordinates. Now we see that an open sentence with two ordered variables not only sorts the set of *ordered pairs* of numbers into two subsets--it also sorts the *points of the plane* into two subsets:

(1) The set of all points whose coordinates satisfy the sentence, and

(2) all other points.

As before, we call this first set of points the *graph* of the sentence.

We shall be interested to learn what sort of figure on the plane this graph will be for any given sentence. Let us try, as an example, the sentence

$$2x - 3y - 6 = 0.$$

We can guess several solutions, such as, (3, 0) and (0, -2). Try to guess some more solutions. Notice that it would be easier to determine solutions if we write an equivalent equation having y by itself on the left side:

$$\begin{aligned} 2x - 3y - 6 &= 0, \\ -3y &= -2x + 6, \\ 3y &= 2x - 6, \\ y &= \frac{2}{3}x - 2. \end{aligned}$$

We call this last equivalent sentence the *y-form* of the original sentence. Now we see that "$y = \frac{2}{3}x - 2$" can be translated into an English sentence in terms of abscissas and ordinates of points on its graph: "The ordinate is 2 less than $\frac{2}{3}$ of the abscissa."

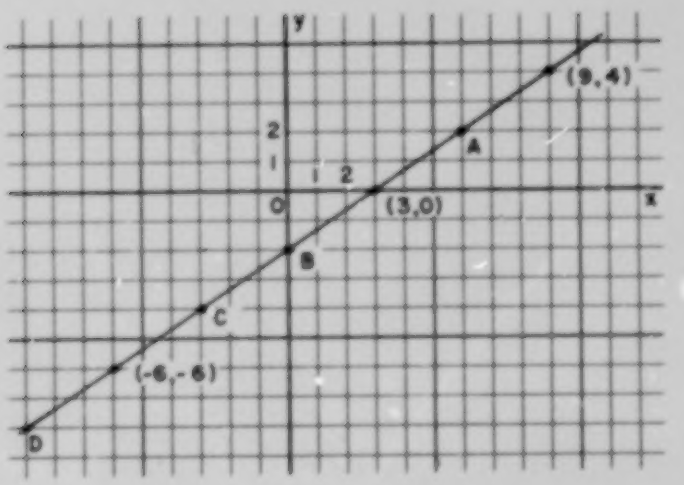


Figure 1.

In Figure 1, the points (-6, -6), (3, 0) and (9, 4) seem to lie on a straight line. This brings up the question: If we draw the line through these points, will we find on it all the points such that each has "ordinate 2 less than $\frac{2}{3}$ of the abscissa"? Furthermore, we must ask: Is every point on this line a point whose "ordinate is 2 less than $\frac{2}{3}$ of the abscissa"?

Suppose we try a point which appears to be on the line, such as point A in Figure 1. The coordinates of this point are (6, 2). Do these coordinates satisfy the equation $2x - 3y - 6 = 0$?

It turns out that every point on this line has coordinates which satisfy the equation $2x - 3y - 6 = 0$.

When we say that a specified line is the graph of a particular open sentence, we mean that both our questions above are answered affirmatively:

(1) if two ordered numbers satisfy the sentence, they are the coordinates of a point on the line;

(2) if a point is on the line, its coordinates satisfy the open sentence.

Thus, the line drawn in Figure 1 is the graph of the sentence

$$2x - 3y - 6 = 0.$$

We can do the same with such open sentences as $3y + 5x - 11 = 0$, $2x + 5 = 0$, $-8y + 1 = 0$, etc., and in each case conclude that the graph is a line.

This suggests that the following general statement is true.

> If an open sentence is of the form
> $$Ax + By + C = 0,$$
> where A, B, C are real numbers with A and B not both zero, then its graph is a line; every line in the plane is the graph of an open sentence of this form.

We know that every open sentence has a graph, and we suspect that every graph is associated with an open sentence. Of course, some open sentences may have graphs with no points (empty graphs) and others with graphs which cover regions of the plane. Later we shall study such sentences.

Problems

1. Where are all the points in the plane whose ordinates are -3?
2. With reference to a set of coordinate axes, find the points, such that:
   (a) each has the abscissa equal to the opposite of the ordinate, using all possible pairs of real numbers which have meaning within the scope of your graph. With reference to the same axes, locate
   (b) the points such that each has ordinate twice the abscissa;
   (c) the points such that each has ordinate that is the opposite of twice the abscissa.

   What general statements can you make concerning these graphs? Write open sentences for each of the graphs drawn.

3. With reference to one set of coordinate axes, draw the graphs of the following and label each one.

(a) $y = x + 5$
(b) $y = x - 3$
(c) $y = 2x + 5$
(d) $y = 2x - 5$
(e) $y = \frac{1}{3}x + 2$
(f) $y = -\frac{1}{3}x - 2$

How does the graph of (a) differ from the graph of (b)? How does the graph of (c) differ from the graph of (d)? How does the graph of (e) differ from the graph of (f)? What is true of the graphs of (a) and (b), and also of the graphs of (c) and (d), that is not true of the graphs of (e) and (f)?

With respect to a set of coordinate axes locate points for which the abscissas are

$$-2,\ -1,\ 0,\ 1,\ 2,\ 3$$

respectively, and for which each ordinate is equal to 3 times the abscissa. Do these points lie on a line?

Now locate points having these same abscissas, but for which each ordinate is greater than 3 times the abscissa. Do these new points lie on a line? Does each one lie above the corresponding point of the first set?

The points in the first set satisfy the sentence

$$y = 3x$$

while those in the second set satisfy the sentence

$$y > 3x.$$

The sentence "$y = 3x$" is the equation of a line, and the graph of "$y > 3x$" is the set of all points above this line, as shown by the shaded portion of Figure 3. Thus, the graph of a sentence such as "$y > 3x$" is the set of all points of the plane for which the sentence is true. If the verb is "is greater than or equal to", that is "$\geq$", we make the boundary line solid, as in Figure 2, while the verb "is greater than" is indicated by using a dashed line for the boundary between the shaded and the unshaded regions as in Figure 3. In these two illustrations, the line is the graph of the sentence $y = 3x$. This graph, which is a line, separates the plane into two half-planes. The graph of $y < 3x$ is the half-plane such that every ordinate is less than three times the abscissa; it is the set of points below the line $y = 3x$. The graph of $y \leq 3x$ is the lower half-plane including the line $y = 3x$.

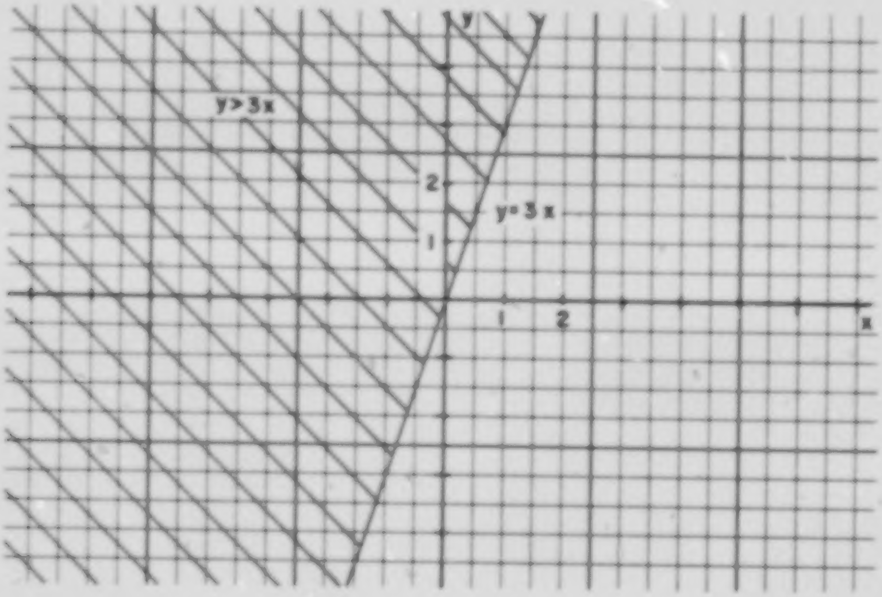


Figure 2. $y \geq 3x$

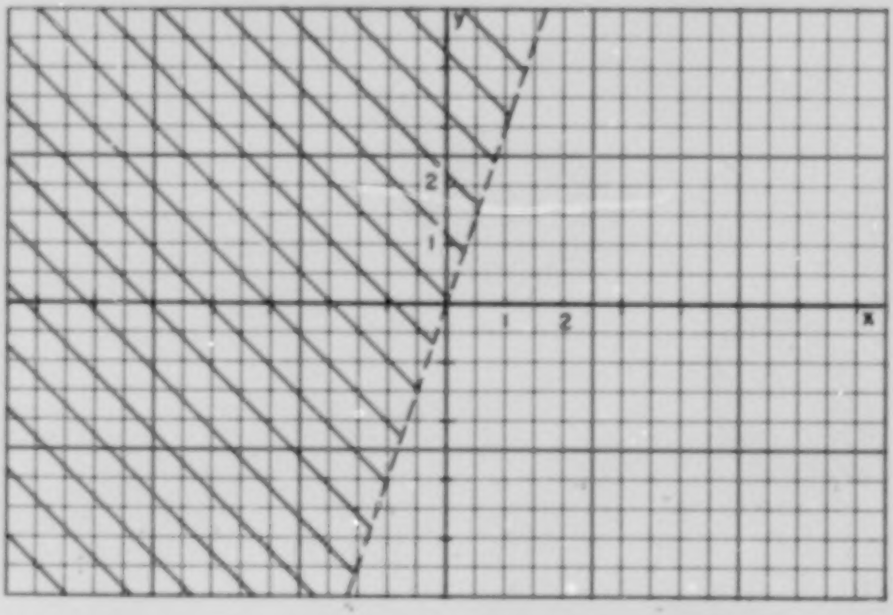


Figure 3. $y > 3x$

Problems

1. With reference to a set of coordinate axes indicate the set of points associated with the ordered pairs of numbers such that each has an ordinate two greater than the abscissa. What open sentence can you write for this set? Now draw the graph of the following open sentences.

   (a) $y > x + 2$; (b) $y \geq x + 2$.

   Is it possible to draw both of these graphs with reference to the same coordinate axes?

2. Given $y = |x|$. In this sentence, is y ever negative? Write the solutions for which the abscissas are: -3, -1, 0, $1\frac{1}{2}$, 2, $\frac{1}{2}$. Draw the graph of the open sentence $y = |x|$ within the confines of your coordinate paper.

In Problem 3

(i) Write the sentence in the y-form.

(ii) Find at least three ordered pairs of numbers which satisfy the equation. (Why do we need no more than two points to graph the line? Another point is desirable as a check.)

(iii) Draw the graph to its full extent on your paper.

3. With reference to one set of axes, draw the graphs of the following.

(a) $3x - 2y = 0$ (d) $3x - 2y = -6$

(b) $3x - 2y = 6$ (e) $3x - 2y = -12$

(c) $3x - 2y = 12$

What is true of the graphs of all these open sentences?

4. Draw the graphs of each of the following with reference to a different set of axes.

(a) $2x - 7y = 14$ (c) $2x - 7y < 14$

(b) $2x - 7y > 14$ (d) $2x - 7y \geq 14$

5. With reference to one set of axes draw the graphs of each of the following.

(a) $5x - 2y = 10$ (c) $5x + y = 10$

(b) $2x + 5y = 10$ (d) $3x - 4y = 6$

Which point seems to lie on three of these lines? Do its coordinates satisfy the open sentences associated with these three lines?

7. Draw the graphs of the open sentences. (Find at least ten ordered pairs satisfying each equation.)

(a) $y = x^2$ (c) $y = x^2 + 1$

(b) $y = -x^2$ (d) $y = \frac{1}{x}$

Are the graphs of these open sentences lines? How do these open sentences differ from those considered in previous problems in this chapter? Can we say that the graph of every open sentence is a straight line?

Graph the following open sentences with reference to the same coordinate axes:

(a) $y = \frac{2}{3}x$ (b) $y = \frac{2}{3}x + 4$ (c) $y = \frac{2}{3}x - 3$

For the first of these no table of values should be necessary. We need simply note that the ordinate must be $\frac{2}{3}$ of the abscissa. In order to get points which are easy to locate we could choose multiples of 3 for values of x. To draw the graph of the second open sentence, we should note that to each ordinate in the graph of the first we add 4. How could we find the ordinates of points for the third open sentence? (Figure 4.)

What are the coordinates of the points at which lines (a), (b), and (c) intersect the vertical axis? Do you see any relation between these points and equations (a), (b), and (c)? We call 0, 4, and -3 the y-intercept numbers of their respective equations. Points (0, 0), (0, 4) and (0, -3) are the y-intercepts of the respective lines. Explain how the graphs of "$y = \frac{2}{3}x + 4$" and "$y = \frac{2}{3}x - 3$" could be obtained by moving the graph of "$y = \frac{2}{3}x$". Notice that the coefficient of x again determines the direction of the lines, whereas the y-intercept numbers determine their positions.

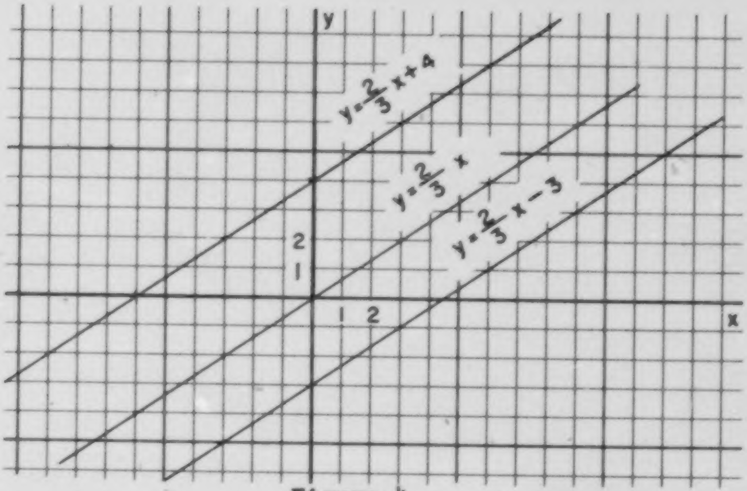


Figure 4.

Write two open sentences such that the absolute value of the y-intercept number is 6 and the coefficient of x is $\frac{2}{3}$. Draw the graphs of these open sentences.

In the figures which you drew in the first part of this section, all of the lines had the same y-intercept, but many different directions. We say that:

> The slope of a line is the coefficient of x in the corresponding sentence written in the y-form. It is the number which determines the direction of the line.

The slope may be either positive, negative or 0. For what positions of the line is the slope negative? 0? What is the slope of the line $x = 2$? Can the equation $x = 2$ be written in y-form? Remember that only non-vertical lines have slopes.

In Figure 5 we have a line which is the graph of "$y = \frac{5}{2}x - 3$". What is the slope of this line? The line passes through points (2, 2) and (4, 7). Verify this. The ordinates of these points are 2 and 7, respectively, and the difference of these ordinates is 7 - 2, or 5. The abscissas of the points are 2 and 4, respectively. The difference of these abscissas is 4 - 2, or 2.

If we divide the difference in ordinates by the difference in abscissas, we obtain the number

$$\frac{7 - 2}{4 - 2} = \frac{5}{2}.$$

But this is the slope of the line! We think of the difference in ordinates as the vertical change and the difference in abscissas as the horizontal change from (2, 2) to (4, 7). Thus,

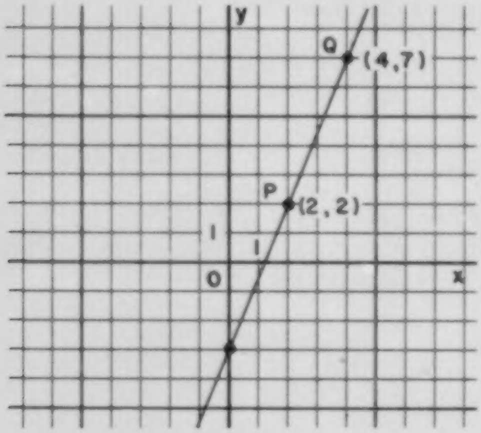


Figure 5.

$$\frac{\text{vertical change}}{\text{horizontal change}} = \frac{7 - 2}{4 - 2} = \frac{5}{2}.$$

Note the order observed in finding these differences: If the first number in the numerator is the ordinate 7 of the point (4, 7), the first number in the denominator must be the abscissa 4 of the same point. What value would we find for the slope if we used as the first number in numerator and denominator the ordinate and abscissa, respectively, of the point (2, 2)? How would this value compare with the value just found?

We can now prove that the ratio of the vertical change to the horizontal change from one point to another on a line will always be the slope of the line.

Theorem. Given two points P and Q on a non-vertical line, the ratio of the vertical change to the horizontal change from P to Q is the slope of the line.

*Proof: Consider the non-vertical line whose equation is

$$Ax + By + C = 0, \quad B \neq 0.$$

(Why must we make the restriction $B \neq 0$?) Let us write this in y-form:

$$y = -\frac{A}{B}x + -\frac{C}{B}.$$

Thus, by definition the slope of this line is $-\frac{A}{B}$. Next, consider two points P and Q on the line with coordinates (c, d) and (a, b), respectively. Since these points are on the line, their coordinates satisfy its equation, giving the true sentences

$$Aa + Bb + C = 0$$
$$Ac + Bd + C = 0.$$

If we subtract the members of these two equations, we have the true sentence

$$A(a - c) + B(b - d) = 0.$$

This may be written as

$$\frac{b - d}{a - c} = -\frac{A}{B}. \qquad \text{(Why?)}$$

But $b - d$ is the vertical change and $a - c$ is the horizontal change from P to Q. This proves the theorem.

What is the slope of the line containing (6, 5) and (-2, -3)? Containing (2, 7) and (7, 3)?

Problems

1. Find the slope of the line through each of the following pairs of points.

   (a) (-7, -3) and (6, 2)

   (b) (-7, 3) and (8, 3)

   (c) (8, 6) and (-4, -1)

   (d) (3, -12) and (-8, 10)

   (e) (4, 11) and (-1, -2)

   (f) (6, 5) and (6, 0)

   (g) (0, 0) and (-6, -2)

   (h) (0, 0) and (-7, 4)

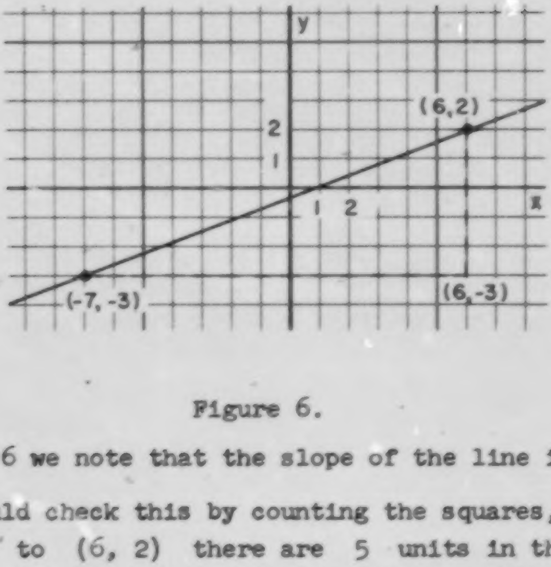


Figure 6.

In Figure 6 we note that the slope of the line is $\frac{2 - (-3)}{6 - (-7)}$ or $\frac{5}{13}$. We could check this by counting the squares, finding that from (-7, -3) to (6, 2) there are 5 units in the vertical change and 13 units in the horizontal change. It would be possible to write the open sentence of this line with a bit more information, that is, if we knew what the y-intercept number is.

In the case of the line in Figure 7 we note that it contains the points $(-6, 6)$ and $(0, -2)$. From this fact we can determine the slope to be $\frac{6 - (-2)}{(-6) - 0} = -\frac{8}{6} = -\frac{4}{3}$. We know, then, that $y = -\frac{4}{3}x$ is the equation of a line with the same slope as the one we have in Figure 7, but which contains the origin. We see that the y-intercept number of the line in Figure 7 is -2; hence, its equation is "$y = -\frac{4}{3}x - 2$". What is the equation for a line parallel to the line in Figure 7, but which contains the point $(0, 6)$?

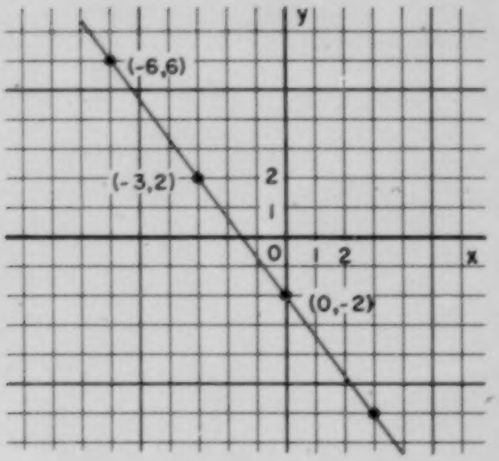


Figure 7.

Problems

1. What is the equation of a line which contains the point $(0, 6)$ and is parallel to the line whose equation is $y = \frac{4}{3}x - 2$?

2. What is the equation of a line parallel to $y = \frac{4}{3}x - 2$ and containing the point $(0, -12)$?

3. What is the slope of all lines parallel to $y = -\frac{4}{3}x$?

4. What is the slope of all lines parallel to $y = -\frac{2}{3}x$?

5. What is the equation of a line whose slope is $-\frac{5}{6}$ and whose y-intercept number is -3?

6. What is the open sentence of a line which passes through (4, 11) and (2, 4) and has y-intercept (0, -3)?

7. What is the equation of the line which contains (5, 6) and (-5, -4) and has y-intercept number 0?

Now let us see how the slope and the y-intercept can help us to draw lines. Suppose a line has slope $-\frac{2}{3}$ and y-intercept number 6. Let us draw the line as well as write its open sentence. To draw the graph, we start at the y-intercept (0, 6). Then we use the slope to locate other points on the line.

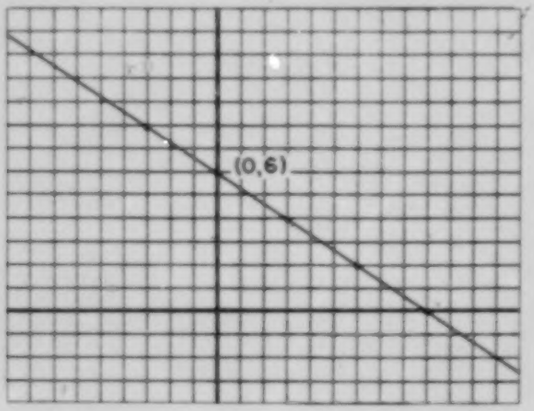


Figure 8.

The fact that the slope is $-\frac{2}{3}$ tells us that between two certain points on the line, the vertical change is -2 while the horizontal change is 3; between two points the vertical change is 4 while the horizontal change is -6; etc. A list of some of the possibilities is

Vertical change	Horizontal change
-2	3
2	-3
4	-6
-10	15

In every case the ratio is $-\frac{2}{3}$. If we take the point which we know is on the line, (0, 6), as one of the two points, we can find another point 3 units to the right and 2 units down; another point is 3 units to the left and 2 units up. We can repeat this process as often as we wish, and quickly get several points through which we may draw the line. Write the open sentence for the line. How would we have chosen the points with respect to (0, 6) if the slope had been $\frac{2}{3}$? What is the open sentence of this line?

What is the form of an equation of a line which has no slope? What is the slope of the line "$x = 2$"? What is an equation of the line through (-3, 4) which has no slope?

Problems

1. With reference to a set of coordinate axes, select the point (-6, -3) and through this point:
   (a) draw the line whose slope is $\frac{5}{6}$. What is an equation of this line?
   (b) Draw the line through (-6, -3) which has no slope. What is an equation of this line?

2. Draw the following lines:

   (a) through the point (2, 1) with slope $-\frac{1}{2}$.

   (b) through the point (3, 4) with slope 0.

   (c) through the point (-3, -4) with no slope. (What type of line has no slope?)

3. Consider the line containing the points (1, -1) and (3, 3). Is the point (-3, -9) on this line? (Hint: Determine the slope of the line containing (1, -1) and (3, 3); then determine the slope of the line containing (1, -1) and (-3, -9).)

4. (a) What do the lines whose open sentences are "$y = \frac{1}{2}x - 3$", "$y = \frac{1}{2}x + 4$", "$y = \frac{2x}{4} - 7$" have in common?

   (b) What do the lines whose open sentences are "$y = \frac{1}{2}x - 3$", "$y = \frac{3x}{4} - 3$", "$y = \frac{7x}{6} - 3$" have in common?

   (c) What do the lines whose open sentences are "$x + 2y = 7$", "$\frac{1}{2}x + y = 3$", and "$2x + 4y = 12$" have in common?

   Show that your answer is correct by drawing the graphs of these three lines.

5. Write each of the following equations in the y-form. Using the slope and the y-intercept, graph each of the lines.

   (a) $2x - y = 7$

   (b) $3x - 4y = 12$

   (c) $4x + 3y = 12$

   (d) $3x - 6y = 12$

   Are you certain that the graphs of these open sentences are lines? Why?

6. Write an equation of each of the following lines.

   (a) The slope is $\frac{2}{3}$ and the y-intercept number is 0.

   (b) The slope is -2 and the y-intercept number is $\frac{4}{3}$.

(c) The slope is -7 and the y-intercept number is -5.

(d) The slope is m and the y-intercept number is b. Can the equation of every straight line be put in this form? What about the equations of the coordinate axes?

7. Write the equation of the line whose y-intercept number is 7 and which contains the point $(6, 8)$. What is the slope of the line? Can you write the slope as $\frac{8 - 7}{6 - 0}$?

8. What is the slope of the line which contains $(-3, 2)$ and $(3, -4)$? If (x, y) is a point on this same line, verify that the slope is also $\frac{y - 2}{x - (-3)}$. Also verify that $\frac{y - (-4)}{x - 3}$ is the slope. If -1 and $\frac{y - 2}{x - (-3)}$ are different names for the slope, show that the equation of the line is "$y - 2 = (-1)(x + 3)$". Show that it can also be written "$y + 4 = (-1)(x - 3)$".

9. Write the equations of the lines through the following pairs of points. (Try to use the method of Problem 8 for parts (c) and (d).)

(a) $(5, 8)$ and $(0, -4)$

(b) $(5, -2)$ and $(0, 6)$

(c) $(-3, 3)$ and $(-5, 3)$

(d) $(4, 2)$ and $(-3, 1)$

10. Any polynomial of first degree in one variable x of the form "$kx + n$", where k and n are real numbers, is said to be linear in x. It is called linear, since the graph of the open sentence "$y = kx + n$" is a straight line. The graph of "$y = kx + n$" is also called the graph of the polynomial "$kx + n$". Draw the graph of each of the following linear polynomials:

(a) $2x - 5$

(b) $-2x + \frac{1}{2}$

(c) $\frac{2}{3}x - 1$

(d) $-\frac{3}{2}x + 2$

11. Consider a circle of diameter d.

(a) Write an expression in d for the circumference of the circle. Is this expression linear in d? What happens to the circumference if the diameter is doubled? Halved? If c is the circumference, what can you say about the ratio $\frac{c}{d}$? How does the value of $\frac{c}{d}$ change when the value of d is changed?

(b) Write an expression in d for the area of the circle. Is this expression linear in d? Is it linear in d^2? If A is the area of the circle, what can you say about the ratio $\frac{A}{d}$? What about the ratio $\frac{A}{d^2}$? Does the value of the ratio $\frac{A}{d}$ change when the value of d is changed? Does the value of $\frac{A}{d^2}$ change when d is changed?

SYSTEMS OF EQUATIONS AND INEQUALITIES

Systems of Equations. We began a study of compound sentences in Part 1. What connectives are used in compound sentences? Let us first consider a compound sentence of two clauses in two variables, whose connective is "or"; for example,

$$x + 2y - 5 = 0 \text{ or } 2x + y - 1 = 0.$$

When is a compound sentence with the connective "or" true? The truth set of this sentence includes all the ordered pairs of numbers which satisfy "$x + 2y - 5 = 0$", as well as all the ordered pairs which satisfy "$2x + y - 1 = 0$", and the graph of its truth set is the pair of lines drawn in Figure 1.

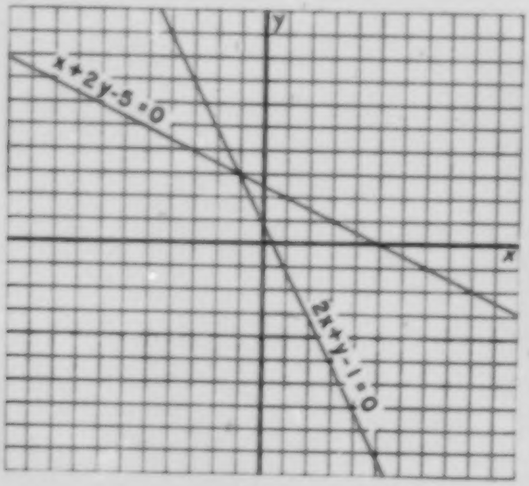


Figure 1.

Name three ordered pairs of numbers which belong to the truth set of

$$x + 2y - 5 = 0.$$

Name four ordered pairs which belong to the truth set of

$$2x + y - 1 = 0.$$

Which of these ordered pairs of numbers are elements of the truth set of the compound open sentence

$$x + 2y - 5 = 0 \text{ or } 2x + y - 1 = 0?$$

If we remember that the sentences "$ab = 0$" and "$a = 0$ or $b = 0$" are equivalent when a and b are real numbers, another way of writing this compound sentence would be

$$(x + 2y - 5)(2x + y - 1) = 0.$$

Now consider a compound sentence with the connective "and" instead of "or". Which ordered pairs are elements of the truth set of the compound open sentence "$x + 2y - 5 = 0$ and $2x + y - 1 = 0$"? Note that only one ordered pair $(-1, 3)$, satisfies both clauses of this sentence, and therefore the graph of the truth set of the open sentence "$x + 2y - 5 = 0$ and $2x + y - 1 = 0$" is the intersection of the pair of lines in Figure 1.

We shall devote most of our attention to compound open sentences made up of two clauses connected by and. This sort of compound open sentence, with the connective "and", is often written

$$\begin{cases} 2x + y - 1 = 0 \\ x + 2y - 5 = 0. \end{cases}$$

This is called a system of equations in two variables. When we talk about the truth set of a system of equations we mean the set of elements common to both the truth sets. As we have seen, the truth set of

$$\begin{cases} 2x + y - 1 = 0 \\ x + 2y - 5 = 0 \end{cases}$$

is $\{(-1, 3)\}$.

Returning to the compound sentence "$x + 2y - 5 = 0$ and $2x + y - 1 = 0$", and looking at Figure 2, we see that there are many compound open sentences whose truth set is $\{(-1, 3)\}$; for example, "$2x + y - 1 = 0$ and $y - 3 = 0$", and "$x + 2y - 5 = 0$ and $x + 1 = 0$" are two such equivalent compound sentences, because their graphs are pairs of lines intersecting in $(-1, 3)$. State at least two more compound sentences whose truth set is $\{(-1, 3)\}$. What is the truth set of

$$x + 1 = 0 \quad \text{and} \quad y - 3 = 0?$$

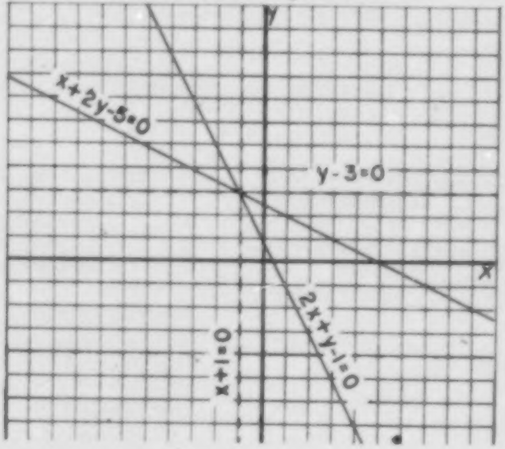


Figure 2.

From this it appears that we could easily find the truth set of any compound open sentence of the type

$$2x + y - 1 = 0 \quad \text{and} \quad x + 2y - 5 = 0$$

if we had a method for getting the equations of the horizontal and vertical lines through the intersection of the graphs of the two clauses.

There are many lines through any point. Here is a method which, as we shall see, will give us the equation of any line through the intersection of two given lines, provided that the lines do intersect in a single point. We shall again use the system

$$\begin{cases} x + 2y - 5 = 0 \\ 2x + y - 1 = 0 \end{cases}$$

to illustrate.

We multiply the expression on the left of the first equation by any number, say 3, and the expression on the left of the second equation by any number, say 7, and form the sentence

$$3(x + 2y - 5) + 7(2x + y - 1) = 0.$$

We see that:

(1) The coordinates of the point of intersection $(-1, 3)$ of the two lines satisfy this new sentence:

$$3\Big(-1 + 2(3) - 5\Big) + 7\Big(2(-1) + 3 - 1\Big) = 3(0) + 7(0) = 0.$$

In general, we know that a point belongs to the graph of a sentence if its coordinates satisfy the sentence. So the graph of our new open sentence

$$"3(x + 2y - 5) + 7(2x + y - 1) = 0"$$

contains the point of intersection of the two lines

$$"x + 2y - 5 = 0" \text{ and } "2x + y - 1 = 0".$$

(2) The graph of the new sentence is a line, because:

$$\begin{aligned} 3(x + 2y - 5) + 7(2x + y - 1) &= 0 \\ 3x + 6y - 15 + 14x + 7y - 7 &= 0 \\ 17x + 13y - 22 &= 0 \end{aligned}$$

and we know that the graph of any equation of the form $Ax + By + C = 0$ is a line, when either A or B is not 0.

Suppose we use this method to find the equation of a line through the intersection of the graphs of the two equations:

$$\begin{cases} 5x - y + 13 = 0 \\ x - 2y - 12 = 0. \end{cases}$$

If we have no particular line in mind, we can use any multipliers we wish. Let us choose 3 and -2, and form the equation:

$$3(5x - y + 13) + (-2)(x - 2y - 12) = 0.$$

Let us assume that the point (c, d) is the point of intersection of the graphs of the two given equations. Since this point (c, d) is on both graphs, we know that

$$5c - d + 13 = 0 \text{ and } c - 2d - 12 = 0$$

is a true sentence. Why?

Substituting the ordered pair (c, d) in the left side of our new equation, we obtain

$$3(5c - d + 13) + (-2)(c - 2d - 12) = 3(0) + (-2)(0) = 0.$$

Hence, we know that if the graphs of the first two equations intersect in a point (c, d), the new line also passes through (c, d), even though we do not know what the point (c, d) is.

In general, we can say:

> If $Ax + By + C = 0$ and $Dx + Ey + F = 0$ are the equations of two lines which intersect in exactly one point, and if a and b are real numbers, then
>
> $$a(Ax + By + C) + b(Dx + Ey + F) = 0$$
>
> is the equation of a line which passes through the point of intersection of the first two lines.

Now that we have a method for finding the equations of lines through the intersection of two given lines, let us see if we can select our multipliers a and b with more care, so that we can get the equations of lines parallel to the axes.

The system

$$\begin{cases} 5x - y + 13 = 0 \\ x - 2y - 12 = 0 \end{cases}$$

gave us some trouble when we tried to guess its truth set from the graph. Let us see if this new approach will help us. Form the sentence

$$a(5x - y + 13) + b(x - 2y - 12) = 0.$$

Let us choose a as 2 and b as -1, so that the coefficients of y become opposites:

$$\begin{aligned} (2)(5x - y + 13) + (-1)(x - 2y - 12) &= 0 \\ 10x - 2y + 26 - x + 2y + 12 &= 0 \\ 9x + 38 &= 0 \\ x + 4\tfrac{2}{9} &= 0. \end{aligned}$$

This last equation represents the line through the intersection of the graphs of the two given equations and parallel to the y-axis. Let us go back and select new multipliers that will give us the equation of the line through the intersection point and parallel to the x-axis. What multipliers shall we use? Since we want the coefficients of x to be opposites we choose $a = 1$ and $b = -5$.

$$\begin{aligned} (1)(5x - y + 13) + (-5)(x - 2y - 12) &= 0 \\ 5x - y + 13 - 5x + 10y + 60 &= 0 \\ 9y + 73 &= 0 \\ y + 8\tfrac{1}{9} &= 0. \end{aligned}$$

We now have the equations of two new lines, "$x + 4\frac{2}{9} = 0$" and "$y + 8\frac{1}{9} = 0$", each of which passes through the point of intersection of the graphs of the first two equations. Why? This reduces our original problem to finding the point of intersection of these new lines. Can you see what it is? We see, then that the truth set of the system

$$\begin{cases} 5x - y + 13 = 0 \\ x - 2y - 12 = 0 \end{cases}$$

is $\{(-4\frac{2}{9}, -8\frac{1}{9})\}$. Verify this fact by showing that these coordinates satisfy both equations.

Now we have a procedure for solving any system of two linear equations. We choose multipliers so as to obtain an equivalent system of lines which are horizontal and vertical. The choice of the multipliers will become easy with practice.

Consider another example: Three times the smaller of two numbers is 6 greater than twice the larger, and three times the larger is 7 greater than four times the smaller. What are the numbers?

The smaller number x and the larger y must satisfy the open sentence

$$3x = 2y + 6 \text{ and } 3y = 4x + 7.$$

This sentence is equivalent to

$$3x - 2y - 6 = 0 \text{ and } 4x - 3y + 7 = 0.$$

Choose multipliers so that the coefficients of x will be opposites. 4 and -3 will do the trick.

$$\begin{aligned} 4(3x - 2y - 6) + (-3)(4x - 3y + 7) &= 0 \\ 12x - 8y - 24 - 12x + 9y - 21 &= 0 \\ y - 45 &= 0. \end{aligned}$$

Now we could choose multipliers so that the coefficients of y would be opposites. Another way to find the line through the intersection and parallel to the y-axis is as follows: On the line "$y - 45 = 0$" every point has ordinate 45. Thus, the ordinate of the point of intersection is 45. The solution of the sentence "$3x - 2y - 6 = 0$" with ordinate 45 is obtained by solving "$3x - 2(45) - 6 = 0$" or its equivalent, "$x - 32 = 0$". Hence, the sentence "$3x - 2y - 6 = 0$ and "$4x - 3y + 7 = 0$" is equivalent to the sentence "$y - 45 = 0$ and $x - 32 = 0$". Now it is easy to read off the solution of the system as $(32, 45)$.

In the above example, what is the solution of the sentence "$4x - 3y + 7 = 0$" with ordinate 45? Does it matter in which sentence we assign the value 45 to y?

Example 1. Find the truth set of

$$\begin{cases} 4x - 3y = 6 \\ 2x + 5y = 16. \end{cases}$$

$$\begin{cases} 4x - 3y - 6 = 0 \\ 2x + 5y - 16 = 0 \end{cases}$$

$$\begin{aligned} 1(4x - 3y - 6) - 2(2x + 5y - 16) &= 0 \\ 4x - 3y - 6 - 4x - 10y + 32 &= 0 \\ -13y + 26 &= 0 \\ 26 &= 13y \\ 2 &= y \end{aligned}$$

When $y = 2$,

$$\begin{aligned} 4x - 3\cdot 2 &= 6 \\ 4x &= 12 \\ x &= 3. \end{aligned}$$

Therefore, "$x = 3$ and $y = 2$" is equivalent to the original sentence.
The truth set is $\{(3, 2)\}$.

Verification:	Left	Right
First clause:	$4\cdot 3 - 3\cdot 2 = 12 - 6$	
	$= 6$	6
Second clause:	$2\cdot 3 + 5\cdot 2 = 6 + 10$	
	$= 16$	16

Example 2. Solve

$$\begin{cases} 3x = 5y + 2 \\ 2x = 6y + 3 \end{cases}$$

$$\begin{cases} 3x - 5y - 2 = 0 \\ 2x - 6y - 3 = 0. \end{cases}$$

$$\begin{aligned} 6(3x - 5y - 2) - 5(2x - 6y - 3) &= 0 \\ 18x - 30y - 12 - 10x + 30y + 15 &= 0 \\ 8x + 3 &= 0 \\ 8x &= -3 \\ x &= -\frac{3}{8} \end{aligned}$$

$$\begin{aligned} 2(3x - 5y - 2) - 3(2x - 6y - 3) &= 0 \\ 6x - 10y - 4 - 6x + 18y + 9 &= 0 \\ 8y + 5 &= 0 \\ 8y &= -5 \\ y &= -\frac{5}{8} \end{aligned}$$

Therefore "$x = -\frac{3}{8}$ and $y = -\frac{5}{8}$" is equivalent to the original sentence.

The solution is $(-\frac{3}{8}, -\frac{5}{8})$.

Verification:	Left	Right
First clause:	$3(-\frac{3}{8}) = -\frac{9}{8}$	$5(-\frac{5}{8}) + 2 = -\frac{25}{8} + \frac{16}{8}$ $= -\frac{9}{8}$
Second clause:	$2(-\frac{3}{8}) = -\frac{3}{4}$	$6(-\frac{5}{8}) + 3 = -\frac{15}{4} + \frac{12}{4}$ $= -\frac{3}{4}$

Problems

1. Find the truth sets of the following systems of equations by the method just developed. Draw the graphs of each pair of equations in (a) and (b) with reference to a different set of axes.

   (a) $\begin{cases} 3x - 2y - 14 = 0 \\ 2x + 3y + 8 = 0 \end{cases}$

   (b) $\begin{cases} 5x + 2y = 4 \\ 3x - 2y = 12 \end{cases}$

   (c) $\begin{cases} 3x - 2y = 27 \\ 2x - 7y = -50 \end{cases}$

   (d) $\begin{cases} x + y - 30 = 0 \\ x - y + 7 = 0 \end{cases}$

   (e) $\begin{cases} 3 - 5x = 0 \\ 3y = x - 6 \end{cases}$

   (f) $\begin{cases} \frac{1}{2}x + y = 2 \\ y - \frac{1}{3}x = 1 \end{cases}$

   (g) $\begin{cases} 7x - 6y = 9 \\ 9x - 8y = 7 \end{cases}$

2. We can also use the operations which yield equivalent open sentences to solve a system of equations. The method which results is essentially the same as that used above. For example, consider the system:

$$\begin{cases} 3x - 2y - 5 = 0 \\ x + 3y - 8 = 0 \end{cases}$$

and assume that (c, d) is a solution of the system. Then each of the following equations is true.

$$3c - 2d - 5 = 0$$
$$c + 3d - 8 = 0$$

$$3(3c - 2d - 5) = 3(0)$$
$$2(c + 3d - 8) = 2(0)$$

$$9c - 6d - 15 = 0$$
$$2c + 6d - 16 = 0$$

$$11c - 31 = 0$$
$$c = \frac{31}{11}$$

Also,

$$3c - 2d - 5 = 0$$
$$-3(c + 3d - 8) = -3(0)$$

$$3c - 2d - 5 = 0$$
$$-3c - 9d + 24 = 0$$

$$-11d + 19 = 0$$
$$d = \frac{19}{11}$$

So _if_ there is a solution of the system

$$\begin{cases} 3x - 2y - 5 = 0 \\ x + 3y - 8 = 0 \end{cases}$$

then that solution is $(\frac{31}{11}, \frac{19}{11})$. We must verify that this is a solution.

$$3(\frac{31}{11}) - 2(\frac{19}{11}) - 5 = 0$$

$$\frac{31}{11} + 3(\frac{19}{11}) - 8 = 0$$

Are these sentences true?

This is often called the _addition_ _method_ of solving systems of equations. Use this method for finding the truth sets of the following systems.

(a) $\begin{cases} x - 4y - 15 = 0 \\ 3x + 5y - 11 = 0 \end{cases}$ (c) $\begin{cases} 2x = 3 - 2y \\ 3y = 4 - 2y \end{cases}$

(b) $\begin{cases} 2x = 3 - 2y \\ 3y = 4 - 2x \end{cases}$ (d) $\begin{cases} 2x = 3 - 2y \\ 3y = 4 - 3x \end{cases}$

3. Translate each of the following into open sentences with two variables. Find the truth set of each.

   (a) Three hundred eleven tickets were sold for a basketball game, some for pupils and some for adults. Pupil tickets sold for 25 cents each and adult tickets for 75 cents each. The total money received was $108.75. How many pupil and how many adult tickets were sold?

   (b) A homeroom bought three-cent and four-cent stamps to mail bulletins to the parents. The total cost was $12.67. If they bought 352 stamps, how many of each kind were there?

   (c) A bank teller has 154 bills of one-dollar and five-dollar denominations. He thinks his total is $465. Has he counted his money correctly?

4. Find the truth sets of the following compound open sentences. Draw the graphs. Do they help you with (b) and (c)?

   (a) $x - 2y + 6 = 0$ and $2x + 3y + 5 = 0$

   (b) $2x - y - 5 = 0$ and $4x - 2y - 10 = 0$

   (c) $2x + y - 4 = 0$ and $2x + y - 2 = 0$

5. Find the equation of the line through the intersection of the lines $5x - 7y - 3 = 0$ and $3x - 6y + 5 = 0$ and passing through the origin. (Hint: What is the value of C so that $Ax + By + C = 0$ is a line through the origin?)

In Problem 4 you found some compound open sentences whose truth sets were not single ordered pairs of numbers. Which ones were they? Let us look more closely at each of them.

In the open sentence

$$"2x - y - 5 = 0 \quad \text{and} \quad 4x - 2y - 10 = 0",$$

we note that "$4x - 2y - 10 = 0$" is equivalent to

$$2(2x - y - 5) = 2(0);$$

so we see that the graphs of both clauses are identical, as shown in Figure 3, and the lines have many points in common.

State some of the numbers of the truth set of the compound sentence. Is the truth set a finite set? What happened when you tried to solve the open sentence algebrically? Why didn't the method work?

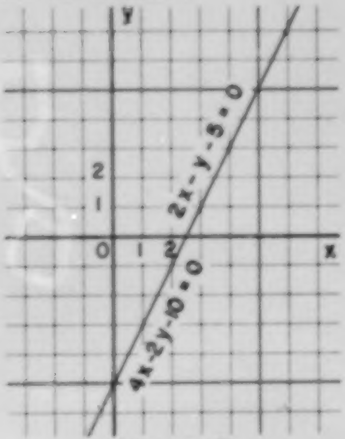


Figure 3.

A somewhat different condition exists in the compound sentence "$2x + y - 4 = 0$ and $2x + y - 2 = 0$". First, write the sentence in the form

$$y = -2x + 4 \quad \text{and} \quad y = -2x + 2.$$

What is the slope of the graph of each of these equations? What is the y-intercept number? We see that the graphs are two parallel lines, as in Figure 4, and there is no intersection point. In such a case, the truth set of the compound sentence is the null set. What happens if we try to solve the sentence algebraically? Why?

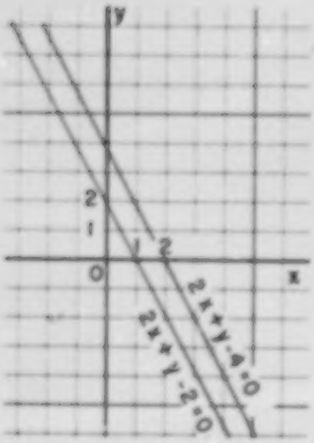


Figure 4.

Let us try to summarize what we have noted here: The truth set of a compound open sentence in two variables, with connective "and", may consist of one ordered pair, many ordered pairs, or no ordered pairs. Correspondingly, the graphs of the two clauses of the open sentence may have one intersection, many intersections, or no intersections.

Example 1.

Equations	Graphs
$2x - 3y = 4$ and $x + y = 7$ $y = \frac{2}{3}x - \frac{4}{3}$ and $y = -x + 7$ The truth set is $\{(5, 2)\}$.	The two lines which are the graphs of the clauses have one intersection, since the slopes of the lines are not the same. The graph of the truth set is the single point (5, 2).

Example 2.

$2x - 3y = 7$ and $4x - 6y = 14$

$y = \frac{2}{3}x - \frac{7}{3}$ and $y = \frac{4}{6}x - \frac{14}{6}$

The truth set is made up of all the ordered pairs whose coordinates satisfy the first equation. (Note that the second clause is obtained if each member of the first clause of the original open sentence is multiplied by 2.)

The graphs of the two clauses of the open sentence coincide, since the lines have the same slope and the same y-intercept number. The entire line is the graph of the truth set.

Example 3.

$2x - 3y = 7$ and $4x - 6y = 3$

$y = \frac{2}{3}x - \frac{7}{3}$ and $y = \frac{4}{6}x - \frac{3}{6}$

The truth set is $\emptyset$.

The graphs of the two clauses of the open sentence are parallel lines, because they have the same slope, but different y-intercept numbers. The graph of the truth set contains no points.

Notice, in Example 3, that the coefficients of x and y in the equation $2x - 3y = 7$ are related in a simple way to those in the equation $4x - 6y = 3$.

$$2 = \frac{1}{2}(4) \quad \text{and} \quad -3 = \frac{1}{2}(-6).$$

In general, the real numbers A and B are said to be proportional to the real numbers C and D if there is a real number k, other than 0, such that

$$A = kC \quad \text{and} \quad B = kD.$$

Thus, the numbers 2 and -3 are proportional to 4 and -6, the number k being $\frac{1}{2}$. If two lines are parallel, what can you say about the coefficients of x and y in their equations?

Another way to say that A and B are proportional to C and D is to say that the ratios

$$\frac{A}{C} \text{ and } \frac{B}{D}$$

are equal, or

$$\frac{A}{C} = \frac{B}{D}.$$

Problems

1. Solve the following compound open sentences.

   (a) $x + 5y - 17 = 0$ and $2x - 3y - 8 = 0$

   (b) $5x - 4y + 2 = 0$ and $10x - 8y + 4 = 0$

   (c) $12x - 4y - 5 = 0$ and $6x + 8y - 5 = 0$

   (d) $x - 2y - 5 = 0$ and $3x - 6y - 12 = 0$

   (e) $\frac{1}{3}(\frac{6x}{7} - \frac{3y}{5}) - 1 = 0$ and $\frac{2}{3}(\frac{4x}{7} + \frac{y}{10}) - \frac{7}{3} = 0$

2. Consider the system,

$$\begin{cases} 2x - y - 7 = 0 \\ 5x + 2y - 4 = 0 \end{cases}$$

Suppose we write these equations in the following form:

$$y = 2x - 7 \text{ and } y = -\frac{5}{2}x + 2$$

(called the "y-form") and draw their graphs.

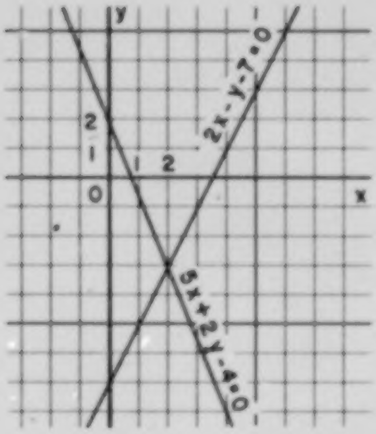


Figure 5.

At what point on the graph of this system are the values of y equal? What is the value of x at this point? If we set the values of y in the two sentences equal to each other, we have the open sentence in one variable,

$$2x - 7 = -\frac{5}{2}x + 2.$$

The truth set of this sentence is $\{2\}$. Using this value for x in each open sentence, we get:

$$y = 2(2) - 7 \qquad\qquad y = -\frac{5}{2}(2) + 2$$

$$y = -3 \qquad\qquad y = -3.$$

Why do we get "$y = -3$" in both cases? Hence, if the compound open sentence "$2x - y - 7 = 0$ and $5x + 2y - 4 = 0$" has a solution, it must be $(2, -3)$. Verify that $(2, -3)$ is the solution.

Suppose that we shorten our work somewhat by writing only the first equation in its y-form.

$$y = 2x - 7$$

Then we replace y in the second equation by the expression "$2x - 7$".

$$5x + 2(2x - 7) - 4 = 0.$$

Let us proceed to solve this open sentence in one variable

$$5x + 4x - 14 - 4 = 0$$
$$9x - 18 = 0$$
$$x = 2.$$

Then,

$$y = 2x - 7 = 2(2) - 7 = -3$$

so that $(2, -3)$ is the possible solution of the system

$$\begin{cases} 2x - y - 7 = 0 \\ 5x + 2y - 4 = 0. \end{cases}$$

The method just described in which we "solve one of the equations for y in terms of x" and then substitute this expression for y into the other equation is called a substitution method. Solve the following systems using whichever of the above methods seems more appropriate.

(a) $\begin{cases} 3x + y + 18 = 0 \\ 2x - 7y - 34 = 0 \end{cases}$ (d) $\begin{cases} 5x + 2y - 5 = 0 \\ x - 3y - 18 = 0 \end{cases}$

(b) $\begin{cases} y = \frac{2}{3}x + 2 \\ y = -\frac{5}{2}x + 40 \end{cases}$ (e) $\begin{cases} x + 7y = 11 \\ x - 3y = -4 \end{cases}$

(c) $\begin{cases} 5x + 2y - 4 = 0 \\ 10x + 4y - 8 = 0 \end{cases}$ (f) $\begin{cases} x = \frac{3}{2}y - 4 \\ y = -\frac{2}{3}x \end{cases}$

3. As we have seen, the truth set of the compound open sentence

$$Ax + By + C = 0 \text{ and } Dx + Ey + F = 0$$

may consist of one ordered pair of numbers, of many ordered pairs, or of no ordered pairs.

(a) If the truth set consists of one ordered pair, what can you say about the graphs of the clauses?

(b) If the truth set consists of many ordered pairs, what is true of the graphs of the two clauses? Are the two clauses of the compound sentence equivalent?

(c) If the truth set is $\emptyset$, how are the coefficients of x and y related in the two clauses? What is true of the graphs of the clauses?

4. Consider the system

$$\begin{cases} 4x + 2y - 11 = 0 \\ 3x - y - 2 = 0. \end{cases}$$

In four different ways find its truth set.

5. Solve in any way. Explain why you chose a particular method in each case.

(a) $\begin{cases} 3x + 2y = 1 \\ 2x - 3y = 18 \end{cases}$

(b) $\begin{cases} x - 2y = 0 \\ x + 2y = 0 \end{cases}$

(c) $\begin{cases} x = 2y - \frac{1}{6} \\ 2x + y = \frac{4}{3} \end{cases}$

(d) $\begin{cases} 3x - 4y - 1 = 0 \\ 7x + 4y - 9 = 0 \end{cases}$

(e) $\begin{cases} \frac{x}{2} - \frac{x}{3} = 1 \\ x + y = 7 \end{cases}$

(f) $\begin{cases} 6y + (2 - 4x) = 3 \\ 4x - 2(3y - 1) = 2 \end{cases}$

(g) $\begin{cases} 5 - (x + y) = 2y \\ 2x - (3y + 1) = 1 \end{cases}$

(h) $\begin{cases} 7x - y = 28 \\ 3x + 11y = 92 \end{cases}$

In Problems 6-13 translate into open sentences, find the truth set and answer the question asked.

6. Find two numbers whose sum is 56 and whose difference is 18.

7. The sum of Sally's and Joe's ages is 30 years. In five years the difference of their ages will be 4 years. What are their ages now?

8. A dealer has some cashew nuts that sell at $1.20 a pound and almonds that sell at $1.50 a pound. How many pounds of each should he put into a mixture of 200 pounds to sell at $1.32 a pound?

9. In a certain two digit number the units' digit is one more than twice the tens' digit. If the units' digit is added to the number, the sum is 35 more than three times the tens' digit. Find the number.

10. Two boys sit on a see-saw, one five feet from the fulcrum (the point where it balances), the other on the other side six feet from the fulcrum. If the sum of the boys' weights is 209 pounds, how much does each boy weigh?

11. Three pounds of apples and four pounds of bananas cost $1.08, while 4 pounds of apples and 3 pounds of bananas cost $1.02. What is the price per pound of apples? Of bananas?

12. A and B are 30 miles apart. If they leave at the same time and walk in the same direction, A overtakes B in 60 hours. If they walk toward each other they meet in 5 hours. What are their speeds?

13. In a 300 mile race the driver of car A gives the driver of car B a start of 25 miles, and still finishes one-half hour sooner. In a second trial, the driver of car A gives the driver of car B a start of 60 miles and loses by 12 minutes. What were the average speeds of cars A and B in miles per hour?

Systems of Inequalities. We have defined a system of equations as a compound open sentence in which two equations are joined by the connective "and". We also introduced a notation for this. Carrying the idea over to inequalities, let us consider systems like the following:

(a) $\begin{cases} x + 2y - 4 > 0 \\ 2x - y - 3 > 0 \end{cases}$ (b) $\begin{cases} 3x - 2y - 5 = 0 \\ x + 3y - 9 \leq 0 \end{cases}$

(c) What would the graph of $x + 2y - 4 > 0$ be? You recall that we first draw the graph of

$$x + 2y - 4 = 0,$$

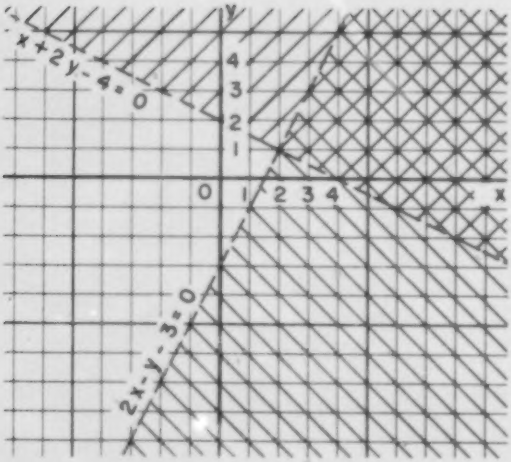


Figure 6.

using a dashed line along the boundary. Why? Then we shade the region above the line, since the graph of "$x + 2y - 4 > 0$", i.e. of "$y > -\frac{1}{2}x + 2$", consists of all those points whose ordinate is greater than "two more than $-\frac{1}{2}$ times the abscissa". In a similar way, we shade the region where "$y < 2x - 3$". This is the region below the line whose equation is "$2x - y - 3 = 0$". Why is the line here also dashed? When would we use a solid line as boundary?

Since the truth set of a compound open sentence with the connective and is the set of elements common to the truth sets of the two clauses, it follows that the truth set of the system (a) is the region indicated by double shading in Figure 6.

(b) What would be the graph of a system in which we have one equation and one inequality, such as Example (b)? What is the graph of "$3x - 2y - 5 = 0$"?

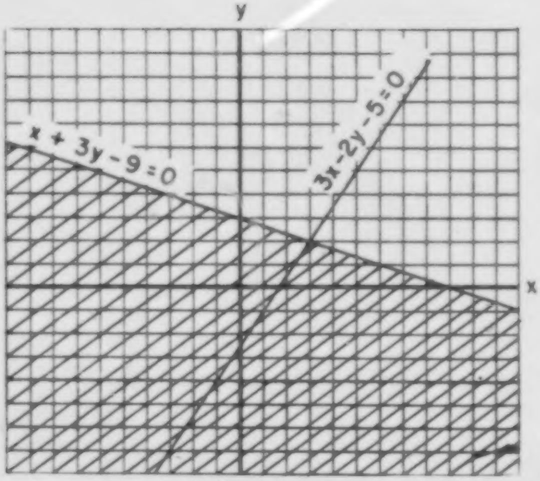


Figure 7.

Is the graph of "$x + 3y - 9 \leq 0$" the region above or below the line

$$x + 3y - 9 = 0?$$

Is the line included? Study Figure 7 carefully, and describe the graph of the system

$$\begin{cases} 3x - 2y - 5 = 0 \\ x + 3y - 9 \leq 0. \end{cases}$$

Problems

Draw graphs of the truth sets of the following systems:

1. $\begin{cases} 2x + y > 8 \\ 4x - 2y \leq 4 \end{cases}$

2. $\begin{cases} 6x + 3y < 0 \\ 4x - y < 6 \end{cases}$

3. $\begin{cases} 5x + 2y + 1 > 0 \\ 3x - y - 6 = 0 \end{cases}$

4. $\begin{cases} 4x + 2y = -1 \\ y - x \geq 4 \end{cases}$

5. $\begin{cases} 2x + y < 4 \\ 2x + y > 6 \end{cases}$

6. $\begin{cases} 2x + y > 4 \\ 2x + y < 6 \end{cases}$

7. $\begin{cases} 2x - y \leq 4 \\ 4x - 2y < 8 \end{cases}$

Let us consider the graph of the compound open sentence

$$x - y - 2 > 0 \quad \text{or} \quad x + y - 2 > 0.$$

First we draw the graphs of the clauses "$x - y - 2 > 0$" and "$x + y - 2 > 0$".

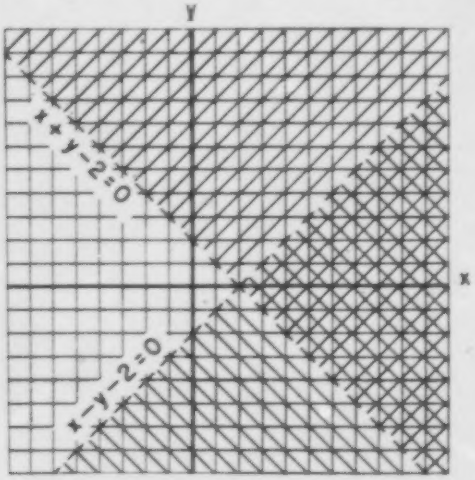


Figure 8.

Next, we recall that the truth set of a compound open sentence with the connective or is the set of all elements in either of the truth sets of the clauses. Hence, the graph of the compound open sentence under consideration includes the entire shaded region in Figure 8.

Problems

Draw the graphs of the truth sets of the following sentences:

1. $2x + y + 3 > 0$ or $3x + y + 1 < 0$
2. $2x + y + 3 < 0$ or $3x - y + 1 < 0$
3. $2x + y + 3 \leq 0$ or $3x + y + 1 \geq 0$
4. $2x + y + 3 > 0$ and $3x - y + 1 < 0$

To complete the picture, let us consider the compound open sentence:

$$(x - y - 2)(x + y - 2) > 0.$$

Remember that "$ab > 0$" means that "the product of a and b is a positive number". What can be said of a and b if $ab > 0$? Thus, we have the two possibilities:

$$x - y - 2 > 0 \text{ and } x + y - 2 > 0,$$

or

$$x - y - 2 < 0 \text{ and } x + y - 2 < 0.$$

In Figure 8, the graph of "$x - y - 2 > 0$ and $x + y - 2 > 0$" is the region indicated by double shading, while the graph of "$x - y - 2 < 0$ and $x + y - 2 < 0$" is the unshaded region. So the graph of

$$(x - y - 2)(x + y - 2) > 0$$

consists of all the points in these two regions of the plane.

Which areas form the graph of the open sentence

$$(x - y - 2)(x + y - 2) < 0?$$

(If $ab < 0$, what can be said of a and b?)

To summarize, we list the following pairs of equivalent sentences (a and b are real numbers):

$ab = 0$: $a = 0$ or $b = 0$.
$ab > 0$: $a > 0$ and $b > 0$, or $a < 0$ and $b < 0$.
$ab < 0$: $a > 0$ and $b < 0$, or $a < 0$ and $b > 0$.

Verify these equivalences by going back to the definition of the product of real numbers.

Problems

1. Draw the graphs of the truth sets of the following open sentences.

   (a) $(2x - y - 2)(3x + y - 3) > 0$

   (b) $(x + 2y - 4)(2x - y - 3) < 0$

   (c) $(x + 2y - 6)(2x + 4y + 4) > 0$

   (d) $(x - y - 3)(3x - 3y - 9) < 0$

2. Draw the graphs of the truth sets of the following open sentences.

   (a) $x - 3y - 6 = 0$ and $3x + y + 2 = 0$

   (b) $(x - 3y - 6)(3x + y + 2) = 0$

   (c) $x - 3y - 6 > 0$ and $3x + y + 2 > 0$

   (d) $x - 3y - 6 < 0$ and $3x + y + 2 < 0$

   (e) $x - 3y - 6 > 0$ and $3x + y + 2 = 0$

   (f) $x - 3y - 6 < 0$ or $3x + y + 2 < 0$

   (g) $x - 3y - 6 = 0$ or $3x + y + 2 \geq 0$

   (h) $(x - 3y - 6)(3x + y + 2) > 0$

   (i) $(x - 3y - 6)(3x + y + 2) < 0$

3. Draw the graph of the truth set of each of these systems of inequalities. (The brace again indicates a compound sentence with connective and.)

   (a) $\begin{cases} x \geq 0 \\ y \geq 0 \\ 3x + 4y \leq 12 \end{cases}$

   (b) $\begin{cases} y \geq 2 \\ 4y \leq 3x + 8 \\ 4y + 5x \leq 40 \end{cases}$

   (c) $\begin{cases} -4 < x < 4 \\ -3 < y < 3 \end{cases}$

4. A football team finds itself on its own 40 yard line, in possession of the ball, with five minutes left in the game. The score is 3 to 0 in favor of the opposing team. The quarterback knows the team should make 3 yards on each running play, but will use 30 seconds per play. He can make 20 yards on a successful pass play, which uses 15 seconds. However, he usually completes only one pass out of three. What combination of plays will assure a victory, or what should be the strategy of the quarterback?

Answers to Problems; page 3:

1. (a) Yes
   (b) Yes
   (c) No
   (d) Yes
   (e) Yes

2. (a) 13
   (b) 23
   (c) $\frac{20}{7}$
   (d) $\frac{4}{3}$
   (e) 2
   (f) 3

Answers to Problems; page 5:

1. (a) False
   (b) True
   (c) False
   (d) True
   (e) False
   (f) True
   (g) False

2. (a) $(3 \times 5) - (2 \times 4) = 7$
   (b) $(3 \times 5 - 2) \times 4 = 52$; or $\big((3 \times 5) - 2\big) \times 4 = 52$
   (c) $(12 \times \frac{1}{2} - \frac{1}{3}) \times 9 = 51$; or $\big((12 \times \frac{1}{2}) - \frac{1}{3}\big) \times 9 = 51$
   (d) $(12 \times \frac{1}{2}) - (\frac{1}{3} \times 9) = 3$
   (e) $12 \times (\frac{1}{2} - \frac{1}{3}) \times 9 = 18$
   (f) $(3 + 4)(6 + 1) = 49$
   (g) $3 + 4(6 + 1) = 31$
   (h) $(3 + 4)6 + 1 = 43$
   (i) $(3 + 4\cdot 6) + 1 = 28$; or $3 + (4\cdot 6 + 1) = 28$

Answers to Problems; pages 13-14:

1. (a) $12(3 + 4) = 12(3) + 12(4)$
   (b) $3(5) + 3(7) = 3(5 + 7)$
   (c) $7(17) + 6(17) = 13(17)$
   (d) $(3 + 11)2 = 3(2) + 11(2)$

2. $(\frac{1}{2} + \frac{2}{3})11 + (\frac{1}{2} + \frac{2}{3})7 = (\frac{1}{2} + \frac{2}{3})(11 + 7)$

$= (\frac{1}{2} + \frac{2}{3})18$

$= \frac{1}{2}(18) + \frac{2}{3}(18)$

$= 9 + 12$

$= 21$

3. (a) $8(\frac{3}{5} + \frac{2}{3}) + (\frac{2}{3} + \frac{3}{5})7 = 8(\frac{3}{5} + \frac{2}{3}) + 7(\frac{3}{5} + \frac{2}{3})$

$= (8 + 7)(\frac{3}{5} + \frac{2}{3})$

$= 15(\frac{3}{5} + \frac{2}{3})$

$= 15(\frac{3}{5}) + 15(\frac{2}{3})$

$= 9 + 10$

$= 19$

(b) $7(\frac{1}{2} + \frac{1}{3} + \frac{3}{4}) + 5(\frac{5}{6} + \frac{3}{4}) = 7\left((\frac{1}{2} + \frac{1}{3}) + \frac{3}{4}\right) + 5(\frac{5}{6} + \frac{3}{4})$

$= 7(\frac{5}{6} + \frac{3}{4}) + 5(\frac{5}{6} + \frac{3}{4})$

$= (7 + 5)(\frac{5}{6} + \frac{3}{4})$

$= 12(\frac{5}{6} + \frac{3}{4})$

$= 12(\frac{5}{6}) + 12(\frac{3}{4})$

$= 10 + 9$

$= 19$

(c) $$\begin{aligned} 5\left(\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5}\right) + 7\left(\tfrac{1}{2} + \tfrac{1}{3}\right) &= 5\left(\left(\tfrac{1}{2} + \tfrac{1}{3}\right) + \tfrac{1}{5}\right) + 7\left(\tfrac{1}{2} + \tfrac{1}{3}\right) \\ &= 5\left(\tfrac{1}{2} + \tfrac{1}{3}\right) + 5\left(\tfrac{1}{5}\right) + 7\left(\tfrac{1}{2} + \tfrac{1}{3}\right) \\ &= 5\left(\tfrac{1}{2} + \tfrac{1}{3}\right) + 7\left(\tfrac{1}{2} + \tfrac{1}{3}\right) + 5\left(\tfrac{1}{5}\right) \\ &= (5 + 7)\left(\tfrac{1}{2} + \tfrac{1}{3}\right) + 5\left(\tfrac{1}{5}\right) \\ &= 12\left(\tfrac{1}{2} + \tfrac{1}{3}\right) + 1 \\ &= 12\left(\tfrac{1}{2}\right) + 12\left(\tfrac{1}{3}\right) + 1 \\ &= 6 + 4 + 1 \\ &= (6 + 4) + 1 \\ &= 11 \end{aligned}$$

Answers to Problems; page 17:

1. (a) The elements of each set are the same: 1, 2, 3, 4, 5, 6.

   (b) The set S = {1, 2, 3, 4, 5, 6} is the set of the first six counting numbers. Many other descriptions are possible.

   (c) The same set may have many descriptions.

2. U = {1, 2, 3, 4}
   T = {1, 4, 9, 16}
   V = {1, 4}; yes. V is a subset of U; yes, V is a subset of T; no, U is not a subset of T, since 2 is not an element of T.

3. K = {1, 2, 3, 4, 9, 16}
   K is not a subset of U; U is a subset of K; T is a subset of K; U is a subset of U (by definition of subset).

4. From the set ∅ one subset, ∅, can be found.

   From the set A = {0}, two subsets, {0} and ∅, can be found.

   From the set B = {0, 1}, four subsets can be found: {0}, {1}, {0, 1}, and ∅.

   From the set C = {0, 1, 2}, eight subsets can be found: {0}, {1}, {2}, {0, 1}, {0, 2}, {1, 2}, {0, 1, 2} and ∅.

   From the set D = {0, 1, 2, 3}, 2^4 or 16 subsets can be found.

   A set with four elements will have twice as many subsets as a set with three elements, and so forth.

Answers to Problems; pages 18-19:

1. (a) Infinite
   (b) Infinite
   (c) Finite
   (d) Finite
   (e) Infinite

2. (a) K = {0}; K is a subset of S; K is a subset of T; S, T, and K are finite.

   (b) M = {0, 2, 4, 5, 6, 7, 8, 9, 10}; M is not a subset of S; T is a subset of M; M is finite.

   (c) R = {5, 7, 9}; R is a subset of both S and of M.

   (d) A cannot be listed; it is the empty set.

   (e) Sets A and K are not the same. A has no elements, while K has one element, 0.

   (f) Yes. A subset of a set can have no elements which are not in the set, so the number of elements in a subset is not greater than the number of elements in the set. Any subset of a finite set, therefore, is finite.

Answers to Problems; pages 23-24:

1. $2(t + 3)$ 2. $\frac{2n + 5}{3}$

3. Both forms are correct. The second is found from the first by use of the associative property for multiplication.

4. $4y$

5. Neither form is correct. $2(a + b)$ and $2a + 2b$ are correct forms.

6. (a) $\frac{39}{2}$
(b) 15
(c) 27
(d) $\frac{9}{4}$
(e) $\frac{3}{2}$
(f) 0
(g) 13
(h) 0
(i) 9
(j) 10

7. (a)
$$\frac{(3a + 4b) - 2c}{3} = \frac{(3\cdot 3 + 4\cdot 2) - 2\cdot 4}{3} = 3$$

(b)
$$\frac{(6a - 4b) + 5c}{5} = \frac{(6\cdot 3 - 4\cdot 2) + 5\cdot 4}{5} = \frac{10 + 20}{5} = 6$$

(c)
$$\frac{(\frac{7a}{2} + \frac{3b}{2}) - \frac{5c}{2}}{2} = \frac{(\frac{7\cdot 3}{2} + \frac{3\cdot 2}{2}) - \frac{5\cdot 4}{2}}{2} = \frac{\frac{1}{2}(21 + 6) - \frac{20}{2}}{2} = \frac{\frac{7}{2}}{2} = \frac{7}{4}$$

$$\text{(d)}\quad \frac{\big(1.5(3) + 3.7(2)\big) - 2.1(4)}{7} = \frac{(4.5 + 7.4) - 8.4}{7}$$

$$= \frac{11.9 - 8.4}{7}$$

$$= \frac{3.5}{7}$$

$$= .5$$

Answers to Problems; page 26:

1. False
2. False
3. True
4. False
5. True
6. True
7. False
8. False
9. False

Answers to Problems; page 27:

1. True when x is 5
2. False when x is 5
3. True when x is 3
   False when x is 4
4. False when x is 4 and y is 3
   True when x is 3 and y is 4
5. False when a is 9 and b is 9
   False when a is 3 and b is 9

Answers to Problems; page 28:

1. 4
2. 14
3. no numbers
4. all numbers
5. all numbers
6. no numbers

Answers to Problems; page 29:

1. (a) 3
   (b) 2
   (c) 1 and 0
   (d) 3
   (e) no value
   (f) 0

2. $\frac{3}{4}$

3. 1.4

Answers to Problems; pages 30-31:

1. (a) $\{\frac{2}{3}\}$
   (b) $\{1, 3\}$
   (c) $\{1, \frac{1}{6}\}$
   (d) $\{1\}$
   (e) $\{0, 2\}$
   (f) $\{4\}$
   (g) $\{\frac{1}{2}\}$
   (h) $\emptyset$

2. One possibility is $x = x + 1$.

Answers to Problems; page 32:

1. $C = \frac{5}{9}(86 - 32)$

   $C = \frac{5}{9}(54)$

   $C = 30$

   C is 30

2. $pv = PV$

   $(15)(600) = 75(V)$

   $(15)(5)(120) = 75(V)$

   $(75)(120) = 75(V)$

   V is 120

3. $A = \frac{1}{2}(B + b)h$

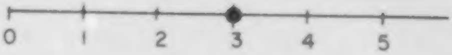
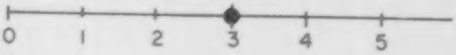
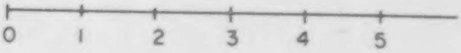
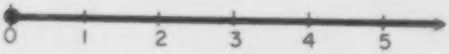
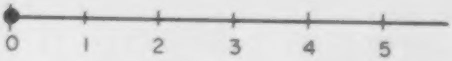
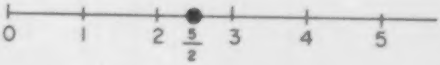
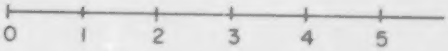
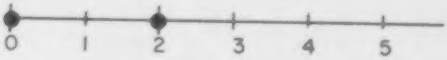
   $20 = \frac{1}{2}(B + 4)4$

   $20 = \frac{1}{2}(4)(B + 4)$

   $20 = 2(B + 4)$

   B is 6

Answers to Problems; page 33:

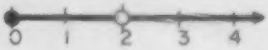
	Truth Set	Graph
1.	$\{3\}$	
2.	$\{3\}$	
3.	$\emptyset$	
4.	all numbers	
5.	$\{0\}$	
6.	$\{\frac{5}{2}\}$	
7.	$\emptyset$	
8.	$\{0, 2\}$	

Answers to Problems; page 34:

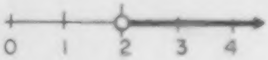
1. False
2. True
3. False
4. True
5. True
6. True
7. True
8. False
9. True
10. False
11. False
12. False

Answers to Problems; page 36:

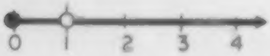
1. (a)



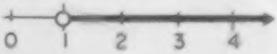
(b)



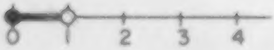
(c)



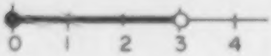
(d)



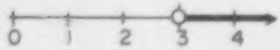
(e)



(f)



(g)



2. (a) $x = 2$

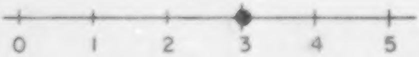
(b) $x < 2$

(c) $x \not< 4$

(d) $x > 1$

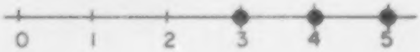
(e) $x \neq 1$

3. (a) {3}

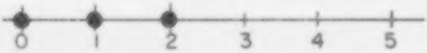


(b) ∅

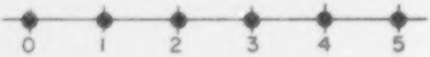
(c) {3, 4, 5}



(d) {0, 1, 2}



(e) {0, 1, 2, 3, 4, 5}



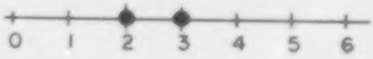
Answers to Problems; page 37:

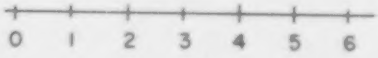
1.	True	4.	True
2.	False	5.	False
3.	False	6.	True

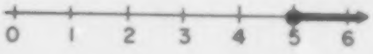
Answers to Problems; page 39:

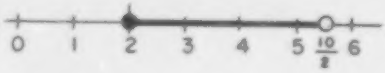
1.	True	4.	False
2.	True	5.	False
3.	False	6.	False

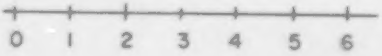
Answers to Problems; page 41:

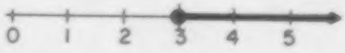
1.


2.


3.


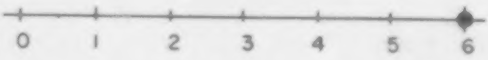
4.

0
1
2
3
4
5
10/2
6

5.


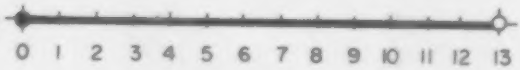
6.


Answers to Problems; page 42:

1. {6}

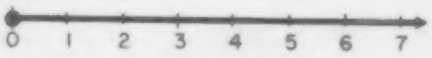


2. all numbers less than 13

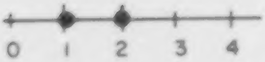

0
1
2
3
4
5
6
7
8
9
10
11
12
13

3. ∅ no graph

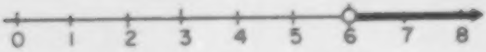
4. all numbers



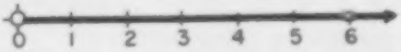
5. {1, 2}



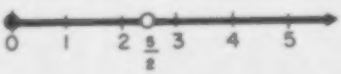
6. all numbers greater than 6



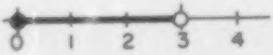
7. all numbers except 0



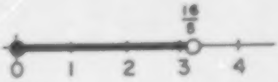
8. all numbers except $\frac{5}{2}$



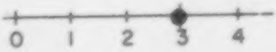
9. all numbers less than 3



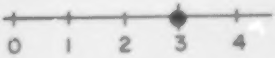
10. all numbers less than $3\frac{1}{5}$



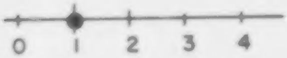
11. {3}



12. {3}



13. {1}



Answers to Problems; page 46:

1. $\frac{1}{2} + \frac{1}{3} = \frac{1}{2}(\frac{3}{3}) + \frac{1}{3}(\frac{2}{2})$

$= \frac{3}{6} + \frac{2}{6}$

$= \frac{5}{6}$

2. $\frac{7}{12} + \frac{5}{18} = \frac{7}{12}(\frac{3}{3}) + \frac{5}{18}(\frac{2}{2})$

$= \frac{21}{36} + \frac{10}{36}$

$= \frac{31}{36}$

3. $$\frac{7+\frac{2}{3}}{\frac{5}{6}} = \frac{7+\frac{2}{3}}{\frac{5}{6}}\left(\frac{6}{6}\right) = \frac{(7+\frac{2}{3})(6)}{(\frac{5}{6})(6)} = \frac{42+4}{5} = \frac{46}{5}$$

4. $$\frac{\frac{1}{2}+\frac{3}{5}}{\frac{3}{20}} = \frac{\frac{1}{2}+\frac{3}{5}}{\frac{3}{20}}\left(\frac{20}{20}\right) = \frac{(\frac{1}{2}+\frac{3}{5})(20)}{(\frac{3}{20})(20)} = \frac{10+12}{3} = \frac{22}{3}$$

5. (a) The other must be zero.
   (b) At least one of the numbers is 0.
   (c) The other property implied here is the converse: if $ab = 0$, then a is 0 or b is 0.

Answers to Problems; pages 49-51:

1. (a) $2m^2n$
   (b) $15p^2q$
   (c) $6n^2m$ (or $6mn^2$)
   (d) $3abc$
   (e) $100ab$
   (f) $36x$

2. (Here we suggest three forms for each. However, there will be other acceptable ones.)

   (a) $(2a)(4b^2)$
   $(8)(ab^2)$
   $(4ab)(2b)$

   (b) $(7)(ab^2)$
   $(7a)(b^2)$
   $(7ab)(b)$

   (c) $(5m)(2n)$
   $(2m)(5n)$
   $(2mn)(5)$

   (d) $(x)(xy^2)$
   $(xy)(xy)$
   $(x^2y)(y)$

   (e) $(64a)(abc^2)$
   $(4a^2)(16bc^2)$
   $(8abc)(8ac)$

   (f) $(2c)(1)$
   $(2)(c)$
   $(1)(2c)$

3. (a) True for every number m--commutative property of multiplication.

   (b) Not true for every number m.

   (c) True for every number a, every number b, and every number y.

   (d) True for every number x and every number y--commutative property of addition.

   (e) True for every number n, every number v, and every number z--associative property of multiplication.

4. (a) A is not closed under addition.
   (b) A is closed under multiplication.

5. (a) S is closed under addition.
   (b) S is closed under multiplication.

6.

If $a \circ b$ is	$2 \circ 6$ is	$\frac{1}{2} \circ 6$ is	$6 \circ 2$ is	$(3 \circ 2) \circ 4$ is
$2a + b$	$2(2) + 6$	$2(\frac{1}{2}) + 6$	$2(6) + 2$	$2\left(2(3) + 2\right) + 4$
$\frac{a + b}{2}$	$\frac{2 + 6}{2}$	$\frac{\frac{1}{2} + 6}{2}$	$\frac{6 + 2}{2}$	$\frac{\frac{3 + 2}{2} + 4}{2}$
$(a - a)b$	$(2 - 2)6$	$(\frac{1}{2} - \frac{1}{2})6$	$(6 - 6)2$	$\left((3-3)2-(3-3)2\right) 4$
$a + \frac{1}{3}b$	$2 + \frac{1}{3}(b)$	$\frac{1}{2} + \frac{1}{3}(3)$	$6 + \frac{1}{3}(2)$	$\left(3+\frac{1}{3}(2)\right) + \frac{1}{3}(4)$
$(a+1)(b+1)$	$(2+1)(6+1)$	$(\frac{1}{2}+1)(6+1)$	$(6+1)(2+1)$	$\left((3+1)(2+1)+1\right)(4+1)$

7. (a) If $a \circ b$ means $\frac{a + b}{2}$, then $b \circ a = \frac{b + a}{2}$.

   $a + b = b + a$ commutative property of addition

   $\frac{a + b}{2} = \frac{b + a}{2}$

   $a \circ b = b \circ a$

   The operation defined here is commutative.

(b) If $a \circ b$ means $(a - a)b$, then $b \circ a$ means $(b - b)a$.

$$0 = 0$$
$$(a - a) = (b - b)$$
$$(a - a)b = (b - b)a$$

The operation defined here is commutative.

(c) If $a \circ b$ means $a + \frac{1}{3}b$, then $b \circ a$ means $b + \frac{1}{3}a$. Then

$$3 \circ 6 = 3 + \tfrac{1}{3}(6)$$

$$6 \circ 3 = 6 + \tfrac{1}{3}(3)$$

But $3 + \frac{1}{3}(b) = 6 + \frac{1}{3}(3)$ is false, so the operation defined here is not commutative.

(d) If $a \circ b$ means $(a + 1)(b + 1)$, then $b \circ a$ means $(b + 1)(a + 1)$.

$$(a + 1)(b + 1) = (b + 1)(a + 1) \quad \text{commutative property of multiplication}$$

$$a \circ b = b \circ a$$

The operation defined here is commutative.

8. (a) If for every a, and every b, $a \circ b = \frac{a + b}{2}$, is $(4 \circ 2) \circ 5 = 4 \circ (2 \circ 5)$ a true sentence?

$$(4 \circ 2) \circ 5 = \frac{\frac{4 + 2}{2} + 5}{2}$$

$$= \frac{3 + 5}{2}$$

$$4 \circ (2 \circ 5) = \frac{4 + \frac{2 + 5}{2}}{2}$$

$$= \frac{4 + \frac{7}{2}}{2}$$

$$= \frac{8 + 7}{4}$$

Since the sentence $\frac{3 + 5}{2} = \frac{8 + 7}{4}$ is false, the operation is not associative.

(b) If for every a and every b, $a \circ b = (a - a)b$, is $(4 \circ 2) \circ 5 = 4 \circ (2 \circ 5)$ a true sentence?

$$\begin{aligned} (4 \circ 2) \circ 5 &= \big((4 - 4)2 - (4 - 4)2\big)5 \\ &= 0 \\ 4 \circ (2 \circ 5) &= (4 - 4)\big((2 - 2)5\big) \\ &= 0 \end{aligned}$$

Since $0 = 0$, the operation is associative for these particular numbers.

We further note that

$$\begin{aligned} (a \circ b) \circ c &= \big((a - a)b - (a - a)b\big)c \\ &= 0 \\ a \circ (b \circ c) &= (a - a)\big((b - b)c\big) \\ &= 0 \end{aligned}$$

So we conclude that the operation is associative.

(c) If for every a and every b, $a \circ b = a + \frac{1}{3}b$, is $(4 \circ 2) \circ 5 = 4 \circ (2 \circ 5)$ a true sentence?

$$\begin{aligned} (4 \circ 2) \circ 5 &= (4 + \tfrac{2}{3}) + \tfrac{5}{3} \\ &= \tfrac{19}{3} \\ 4 \circ (2 \circ 5) &= 4 + (\tfrac{1}{3})(2 + \tfrac{5}{3}) \\ &= 4 + \tfrac{1}{3}(\tfrac{11}{3}) \\ &= 4 + \tfrac{11}{9} \end{aligned}$$

Since $\frac{19}{3} = 4 + \frac{11}{9}$ is false, the operation is not associative.

(d) If for every a and every b, $a \circ b = (a + 1)(b + 1)$, is $(4 \circ 2) \circ 5 = 4 \circ (2 \circ 5)$ a true sentence?

$$\begin{aligned}(4 \circ 2) \circ 5 &= \big((4 + 1)(2 + 1) + 1\big)(5 + 1)\\ &= (16)(6)\\ 4 \circ (2 \circ 5) &= (4 + 1)\big((2 + 1)(5 + 1) + 1\big)\\ &= (5)(19)\end{aligned}$$

Since $(16)(6) = (5)(19)$ is false, the operation is not associative.

Answers to Problems; pages 53-54:

1. (a) $6r + 6s$ (d) $7x + x^2$
   (b) $ba + 3a$ (e) $48 + 30$
   (c) $x^2 + xz$ (f) $ab + b^2$

2. (a) $3(x + y)$ (d) $\frac{1}{2}(x + y)$
   (b) $a(m + n)$ (e) $(2 + a)a$
   (c) $(1 + b)x$ (f) $x(x + y)$

3. (a) $$\begin{aligned}14x + 3x &= (14 + 3)x\\ &= 17x\end{aligned}$$

   (b) $$\begin{aligned}\tfrac{3}{4}x + \tfrac{3}{2}x &= (\tfrac{3}{4} + \tfrac{3}{2})x\\ &= (\tfrac{3}{4} + \tfrac{6}{4})x\\ &= \tfrac{9}{4}x\end{aligned}$$

(c) $\frac{2}{3}a + 3b + \frac{1}{3}a = \frac{2}{3}a + \frac{1}{3}a + 3b$

$= (\frac{2}{3} + \frac{1}{3})a + 3b$

$= 1a + 3b$

$= a + 3b$

(d) $7x + 13y + 2x + 3y = 7x + 2x + 13y + 3y$

$= (7 + 2)x + (13 + 3)y$

$= 9x + 16y$

(e) $4x + 2y + 2 + 3x = 4x + 3x + 2y + 2$

$= (4 + 3)x + 2y + 2$

$= 7x + 2y + 2$

(f) $1.3x + 3.7y + 6.2 + 7.7x = 1.3x + 7.7x + 3.7y + 6.2$

$= (1.3 + 7.7)x + 3.7y + 6.2$

$= 9.0x + 3.7y + 6.2$

(g) $2a + \frac{1}{3}b + 5$--no simpler form.

Answers to Problems; pages 55-56:

1. (a) $6m + 3pm$ (d) $2x^2 + x^2y$
   (b) $2k^2 + 2k$ (e) $eh + fh + gh$
   (c) $12s + 18r + 42q$ (f) $6p^2q + 6pq^2$

2. (a) is false for every number a and every number b.
   (b) is true for every number x and every number y.
   (c) is true for every number a, every number b, and every number c.
   (d) is not true for every number a, every number b, and every number c.
   (e) is not true for every number x.
   (f) is true for every number x and every number y.

3. (a) $(3u + v)v$ (d) $(1 + 2d)(2c)$

(b) $7q(p + r)$ (e) $(1 + 2x)(3x)$

(c) $(1 + x)3x$ (f) $xz(z + 2)$

Answers to Problems; pages 57-58:

1. $$\begin{aligned}(x + 4)(x + 2) &= (x + 4)x + (x + 4)2\\ &= x^2 + 4x + 2x + 8\\ &= x^2 + (4 + 2)x + 8\\ &= x^2 + 6x + 8\end{aligned}$$

2. $$\begin{aligned}(x + 1)(x + 5) &= (x + 1)x + (x + 1)5\\ &= x^2 + 1x + 5x + 5\\ &= x^2 + (1 + 5)x + 5\\ &= x^2 + 6x + 5\end{aligned}$$

3. $$\begin{aligned}(x + a)(x + 3) &= (x + a)x + (x + a)3\\ &= x^2 + ax + 3x + 3a\end{aligned}$$

4. $$\begin{aligned}(x + 2)(y + 7) &= (x + 2)y + (x + 2)7\\ &= xy + 2y + 7x + 14\end{aligned}$$

5. $$\begin{aligned}(m + n)(m + n) &= (m + n)m + (m + n)n\\ &= m^2 + mn + mn + n^2\\ &= m^2 + 1mn + 1mn + n^2\\ &= m^2 + (1 + 1)mn + n^2\\ &= m^2 + 2mn + n^2\end{aligned}$$

6. $$\begin{aligned}(2p + q)(p + 2q) &= (2p + q)p + (2p + q)2q\\ &= 2p^2 + pq + 4pq + 2q^2\\ &= 2p^2 + (1 + 4)pq + 2q^2\\ &= 2p^2 + 5pq + 2q^2\end{aligned}$$

7. See Text.

8. $19 \times 13 = 19(10 + 3)$

$= 19(10) + 19(3)$ - distributive property

$= 19(10) + (10 + 9)3$

$= 19(10) + \big(10(3) + 9(3)\big)$ - distributive property

$= \big(19(10) + 10(3)\big) + 9(3)$ - associative property of addition

$= (19 + 3)10 + 9(3)$ - distributive property

$15 \times 14 = (15 + 4)10 + 20$

$13 \times 17 = (13 + 7)10 + 21$

$11 \times 12 = (11 + 2)10 + 2$

9. Since $\frac{3}{4}$ is $\frac{18}{24}$ and $\frac{5}{6}$ is $\frac{20}{24}$, the coordinate of one point between the two is $\frac{19}{24}$. There are infinitely many points between $\frac{3}{4}$ and $\frac{5}{6}$.

10. Set T is closed under addition, since the sum of two elements gives an integral multiple of 3. Set T is not closed under "averaging", since the average of two elements (such as 3 and 6) is not necessarily an integer.

11. (a) $\emptyset$

(b) $\{0\}$

(c) 0 and $\frac{1}{2}$ and all numbers between 0 and $\frac{1}{2}$

12. $$\frac{\frac{3}{5}+\frac{2}{3}}{\frac{3}{4}} = \frac{\frac{3}{5}+\frac{2}{3}}{\frac{3}{4}}\left(\frac{60}{60}\right)$$

$$= \frac{(\frac{3}{5}+\frac{2}{3})(60)}{(\frac{3}{4})(60)}$$

$$= \frac{36+40}{45}$$

$$= \frac{76}{45}$$

13. $3x + y + 2x + 3y = (3x + 2x) + (y + 3y)$ by the associative and commutative properties of addition

$= (3 + 2)x + (1 + 3)y$ by the distributive property

$= 5x + 4y$

Since the associative, commutative, and distributive properties are true for all numbers,

$$3x + y + 2x + 3y = 5x + 4y$$

for all numbers.

14. (a) $(x + 1)(x + 1) = (x + 1)x + (x + 1)1$ - distributive

$= x^2 + x + x + 1$ - distributive

$= x^2 + (1 + 1)x + 1$ - distributive

$= x^2 + 2x + 1$

$(x + 2)(x + 2) = x^2 + 4x + 4$ - as above.

(b) $x^2 + 6x + 9 = x^2 + (3 + 3)x + 9$

$= x^2 + 3x + 3x + 9$ - distributive

$=(x + 3)x + (x + 3)3$ - distributive

Since $x + 3$ is a number by closure

$=(x + 3)(x + 3)$ - distributive.

Answers to Problems; pages 60-62:

1. See text.

2. If Jonathan weighs n pounds, the phrase is: The number of pounds in the weight of a boy who is 7 pounds heavier than Jonathan.

3. If I am n years old now, the phrase is: The number of years in my age 7 years ago.

4. If there are y students in the class, the phrase is: Half of the students in the class.

5. If rhubarb costs r cents per bunch and a shopping bag costs 5 cents, the phrase is: The cost in cents of 2 bunches of rhubarb and a shopping bag to put them in.

6. If a coal mine produced a tons of coal one day and b tons of coal the next day, the phrase is: The number of tons of coal produced in the two days. (Notice that we are using two variables here. While it will be some time before the students do much with more than one variable, we need not shut our eyes to them.)

7. See text.

8. If f is the number of feet, then the phrase is $12f$.

9. If k is the number of quarts, then the phrase is $2k$.

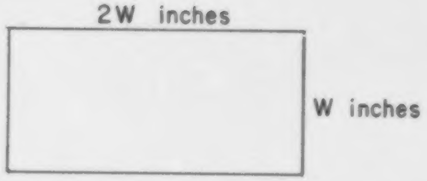
10. If n is the whole number, then the phrase is $n + 1$.

11. If n is the number, then the phrase is $\frac{1}{n}$.

12. If there are k pounds and t ounces, then the phrase is $16k + t$.

13. If there are m dollars, k quarters, m dimes, and n nickels, then the phrase is $100m + 25k + 10m + 5n$. (Notice that m represents the number of dimes as well as the number of dollars. This means that we have the same number of dimes as dollars, since a variable cannot represent two different numbers at the same instant. The open phrase might be written $110m + 25k + 5n$.)

14.



If the rectangle is w inches wide, then the phrase is $2w$.

15. If one side of the square is a feet long, then the phrase is $4a$.

16. (a) $\frac{1}{2}(x + y)$ (b) $x + y + \frac{1}{2}(x + y)$

17. If a people bought tickets, then the phrase is $2a$.

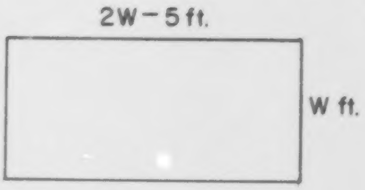
18. $\frac{1}{d}$ An inductive approach may help the reader to see some of these. Here he will probably respond to the question, "How much of the house will he paint in one day if it takes him 8 days to paint the house"?

19. $x \cdot \frac{1}{5}$ or $\frac{x}{5}$

20. $r + 12$

21. If the plant grows g inches per week, then the phrase is $20 + 5g$.

22. If the rectangle is w feet wide, then the phrase is



(a) $2w - 5$

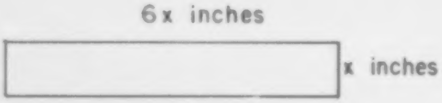
(b) $w + (2w - 5) + w + (2w - 5)$
or $2w + 2(2w - 5)$

(c) $w(2w - 5)$

In the early part of our work with translation we have been trying to emphasize the idea that the variable represents a number by being reasonably precise in the language. Thus, we have been saying, "the number of feet in the width", "the number of inches in the base", etc. As we go on, we become more careless about this way of speaking in order to be able to speak more fluently. So in Problem 22, as long as the unit of length and the unit of area are clearly stated elsewhere in the problem, we allow ourselves to speak of "perimeter" and "area" instead of "number of feet in the perimeter" and "number of square feet in the area".

Answers to Problems; pages 64-65:

1. If the rectangle is x inches wide, then it is $6x$ inches long, and $x + 6x + x + 6x = 144$ or $2x + 2(6x) = 144$.



2. If the smallest angle is s degrees, then the largest angle is $2s + 20$ degrees, and $s + (2s + 20) + 70 = 180$.

3. If there were y students in Miss. Jones's class, then there were $y + 5$ students in Mr. Smith's class and $y + (y + 5) = 43$.

   If there were y students in Miss. Jones's class, then there were $43 - y$ students in Mr. Smith's class and $43 - y = y + 5$.

   (Call attention to the second of these methods. It is a useful approach to know, but one with which some readers have trouble at first.)

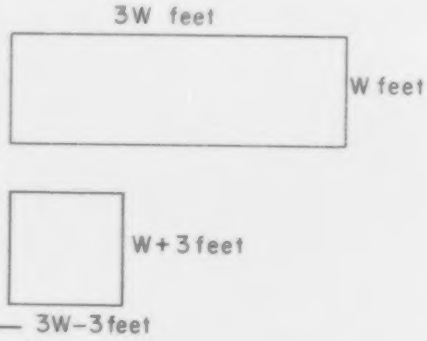
4. If Dick is x years old now, then John is $3x$ years old now. Dick was $x - 3$ years old three years ago, John was $3x - 3$ years old three years ago, and $(x - 3) + (3x - 3) = 22$.

5. If John has d dimes, then he has $d + 1$ quarters, he has $2d + 1$ nickels, his dimes are worth $10d$ cents, his quarters are worth $25(d + 1)$ cents, his nickels are worth $5(2d + 1)$ cents and
$10d + 25(d + 1) + 5(2d + 1) = 165$.

6. If the freight train traveled z miles per hour, then the passenger train traveled $z + 20$ miles per hour, the freight train went $5z$ miles, the passenger train went $5(z + 20)$ miles, and $5(z + 20) = 5z + 100$.

7. If y is the number of years until the men earn the same salary, then
Mr. Brown will be earning $3600 + 300y$ dollars,
Mr. White will be earning $4500 + 200y$ dollars, and
$3600 + 300y = 4500 + 200y$.

8. If the table is w feet wide, then it is $3w$ feet long, the square would be $w + 3$ feet wide and $3w - 3$ feet long, and $w + 3 = 3w - 3$.

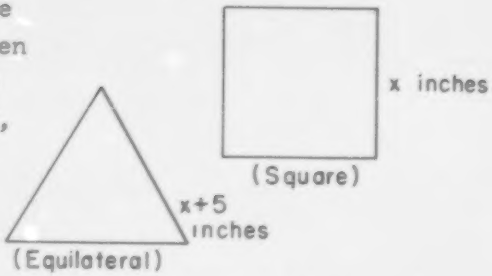


Answers to Problems; pages 67-68:

1. If the number is n, then $\frac{3}{4}n + \frac{1}{3}n \geq 26$.

2. If Norman is y years old, then Bill is $y + 5$ years old, and $y + (y + 5) < 23$.

3. If a side of the square is x inches long, then a side of the triangle is $x + 5$ inches long, and $4x = 3(x + 5)$.



4. If the rate of the current is c miles per hour, then $c + 12$ is the rate of the boat downstream, and $c + 12 < 30$.

5. If there are c minutes allowed for commercials, then $c \geq 3$ and $c < 12$.
Time for material other than advertising is $30 - c$.
Another way of expressing the last idea is $T \leq 30 - 3$ and $T > 30 - 12$.

6. If his score on the third test is t, then

$$\frac{75 + 82 + t}{3} \geq 88.$$

$$\frac{75 + 82 + 100}{3} = \frac{257}{3} = 85\frac{2}{3}$$

$$\frac{75 + 82 + 0}{3} = \frac{157}{3} = 52\frac{1}{3}$$

7. If there are s students in Scott School and m students in Morris School, then

(a) $s > m$

(b) $s = m + 500$.

Answers to Review Problems; pages 68-71:

1. (a) If the whole number is n, then its successor is $n + 1$, and $n + (n + 1) = 575$.

   (b) If the whole numbers is n, then its successor is $n + 1$, and $n + (n + 1) = 576$.
   This sentence is false for all whole numbers.
   If a number is odd, its successor is even; if the number is even, its successor is odd.
   Hence, their sum cannot be even.

   (c) If the first number is n, then the second number is $n + 1$, and $n + (n + 1) = 576$.
   There is a number for which this sentence is true, since the domain of the variable is not restricted to whole numbers.

   (d) If one piece of the board is f feet long, then the other piece is $2f + 1$ feet long, and $f + (2f + 1) = 16$.

   (e) $3x = 225$.

2. If the tens' digit is t, and the units' digit is u, then the number is $10t + u$, and $10t + u = 3(t + u) + 7$.

3. $42 - n$

4. (a) If n is the number, then $3(n + 17) = 192$.

   (b) If n is the number, then $3(n + 17) < 192$.

5. If the first number is x, then the second number is $5x$, and $x + 5x = 4x + 15$.

6. In one hour he can plow $\frac{1}{7}$ of the field with the first tractor. In two hours, using both tractors, he can plow $\frac{2}{7} + \frac{2}{5}$ of the field. The part of the field left unplowed is $1 - (\frac{2}{7} + \frac{2}{5})$. The open sentence is $\frac{x}{7} + \frac{x}{5} = 1$.

7. Mr. Brown's weight m months ago was $175 + 5m$. $175 + 5m = 200$.

8. (a) If n is a whole number, then $n + 1$ is its successor, and $n + (n + 1) = 45$.

   (b) If n is an odd number, then $n + 2$ is the next consecutive odd number, and $n + (n + 2) = 75$. See whether your students notice that this can never be true for any odd number, since the sum of two odd numbers is even.

9. If the marked price was m dollars, then $176 = m - \frac{12}{100}m$.

10. If x is the number of dollars for one hour's work at the normal rate, then $\frac{3}{2}x$ is the number of dollars for one hour's work at the over-time rate, and $40x + 8(\frac{3}{2}x) = 166.40$.

11. (a) $35 + 20t$

    (b) $h - 1$ is the number of one-hour periods after the initial hour, and the phrase is $35 + 20(h - 1)$.

12. If the radiator originally contains w quarts of water, it contains $w + 2$ quarts of mixture after the alcohol was added. Since 20% of this mixture is alcohol, there are $\frac{20}{100}(w + 2)$ quarts of alcohol in the mixture. $2 = \frac{20}{100}(w + 2)$.

13. (a) $100x + 40y$

    (b) $100(2 \times 60) + 40y$

    (c) $100x + 40y = 20,000$

14. This problem reviews sums of pairs of members of a set; it is not a problem set up primarily to get an answer. The pupil who tries to write an open sentence will find he is wasting his time. Instead he should observe that the man has a set of four members: {1.69, 1.95, 2.65. 3.15} and that he should examine the set of all possible sums of pairs of elements of the set.

+	1.69	1.95	2.65	3.15
1.69	3.38	3.64	4.34	4.84
1.95	3.64	3.90	4.60	5.10
2.65	4.34	4.60	5.30	5.80
3.15	4.84	5.10	5.80	6.30

This is the set: {3.38, 3.64, 3.90, 4.34, 4.60, 4.84, 5.10, 5.30, 5.80, 6.30}

From this we see:

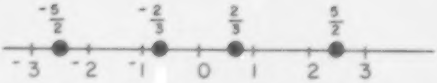
(a) The smallest amount of change he could have is 5.00 - 4.84, or 16 cents.

(b) The greatest amount of change possible is 5.00 - 3.38, or $1.62.

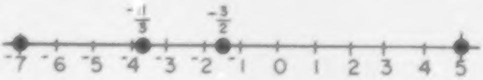
(c) There are four pairs of two boxes he cannot afford: one of $1.95 and one of $3.15; one of $2.65 and one of $3.15; two of $2.65; two of $3.15.

Answers to Problems; pages 76-77:

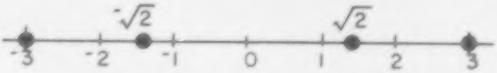
1. (a) $\{^-(\frac{2}{3}), \frac{2}{3}, \frac{5}{2}, ^-(\frac{5}{2})\}$


$^-\frac{5}{2}$
$^-\frac{2}{3}$
$\frac{2}{3}$
$\frac{5}{2}$
$^-3$ $^-2$ $^-1$ 0 1 2 3

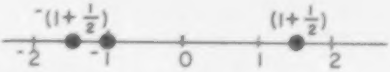
(b) $\{^-(\frac{3}{2}), 5, ^-7, ^-(\frac{11}{3})\}$


$^-\frac{11}{3}$
$^-\frac{3}{2}$
$^-7$ $^-6$ $^-5$ $^-4$ $^-3$ $^-2$ $^-1$ 0 1 2 3 4 5

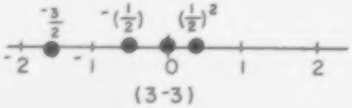
(c) $\{\sqrt{2}, ^-\sqrt{2}, 3, ^-3\}$


$^-\sqrt{2}$
$\sqrt{2}$
$^-3$ $^-2$ $^-1$ 0 1 2 3

(d) $\{^-1, ^-(1+\frac{1}{2}), (1+\frac{1}{2})\}$


$^-(1+\frac{1}{2})$
$(1+\frac{1}{2})$
$^-2$ $^-1$ 0 1 2

(e) $\{^-(\frac{1}{2}), (\frac{1}{2})^2, ^-(\frac{6}{4}), (3-3)\}$


$^-\frac{3}{2}$
$^-(\frac{1}{2})$
$(\frac{1}{2})^2$
$^-2$ $^-1$ 0 1 2
(3-3)

2. (a) 0 is to the right of $^-(\frac{5}{2})$.

(b) $^-(\frac{5}{2})$ and $^-(\frac{10}{4})$ are names for the same number and, so, name the same point on the number line.

(c) 3 is to the right of 0.

(d) $\sqrt{2}$ is to the right of $^-4$.

(e) $^-(\frac{21}{4})$ is to the right of $^-(\frac{16}{3})$.

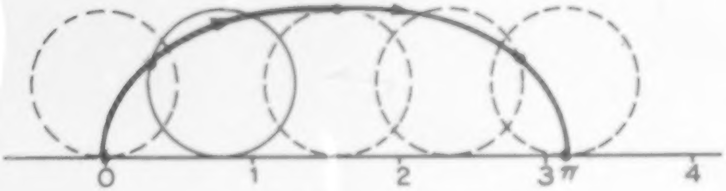
$^-(\frac{21}{4} \times \frac{3}{3}) = {}^-(\frac{63}{12})$ and $^-(\frac{16}{3} \times \frac{4}{4}) = {}^-(\frac{64}{12})$.

$^-(\frac{63}{12})$ is to the right of $^-(\frac{64}{12})$.

Because of the manner in which the negative numbers were constructed on the number line, the negative number, $^-(\frac{63}{12})$, corresponding to the lesser of the two numbers of arithmetic, is to the right of $^-(\frac{64}{12})$.

(f) $\frac{1}{2}$ is to the right of $^-(\frac{1}{2})$.

3.



Since "rolling the circle" gives us a length on the number line equal to the circumference of the circle, the circle comes to rest at a point on the line π units to the right of 0. If the circle is rolled to the left one revolution, the circle will come to rest at a point on the number line whose coordinate is $^{-}\pi$.

4. (a) $^{-}2$ is an integer, a rational number, a real number.

   (b) $^{-}(\frac{10}{3})$ is a rational number and a real number.

   (c) $^{-}\sqrt{2}$ is a real number.

5. A = {0, 1, 2, 3, ...}

   B = {1, 2, 3, 4, ...}

   C = {0, 1, 2, 3, ...}

   I = {..., -3, -2, -1, 0, 1, 2, 3, ...}

   N = {1, 2, 3, ...}

   A, the set of whole numbers, and C, the set of non-negative integers, are the same since both are comprised of the positive integers and 0. B, the set of positive integers, and N, the set of counting numbers, are the same, since both consist of the counting numbers, 1, 2, 3, ...

Answers to Problems; pages 78-80:

1. (a) $3 \leq {}^-1$ is false, since 3 is to the right of ${}^-1$ on the number line and is therefore greater than ${}^-1$. Note again the easy comparison of a positive number and a negative number.

   (b) $2 < {}^-(\frac{7}{2})$ is false, since 2 is to the right of ${}^-(\frac{7}{2})$ and is therefore greater than ${}^-(\frac{7}{2})$.

   (c) ${}^-4 \nless 3.5$ is false.

   (d) ${}^-(\frac{12}{5}) < {}^-2.2$ is true. Changing the decimal fraction to a common fraction, the statement becomes ${}^-(\frac{12}{5}) < {}^-(\frac{22}{10})$. Now ${}^-(\frac{12}{5}) = {}^-(\frac{24}{10})$ and, so, is to the left of ${}^-(\frac{22}{10})$ on the number line. Thus, ${}^-(\frac{12}{5})$ is to the left of ${}^-2.2$.

   (e) ${}^-(\frac{3}{5}) \geq {}^-(\frac{3 + 0}{5})$ is true, since ${}^-(\frac{3}{5})$ and ${}^-(\frac{3 + 0}{5})$ are names for the same number, they correspond to the same point on the number line. But any real number is greater than or equal to itself!

   The truth status of sentences (f) - (j) can be determined in the same manner as (a) and (b) above; consequently, we simply list the results.

   (f) ${}^-4 \ngtr 3.5$ is true.

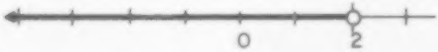
   (g) ${}^-6 > {}^-3$ is false.

   (h) $3.5 < {}^-4$ is false.

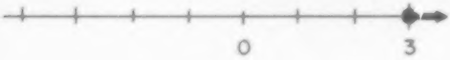
   (i) ${}^-3 < {}^-2.8$ is true.

   (j) ${}^-\pi \nless {}^-2.8$ is false.

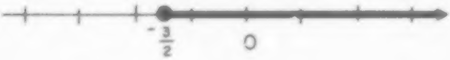
2. (a) $y < 2$.



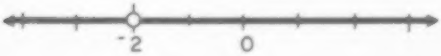
(b) $u \nless 3$.



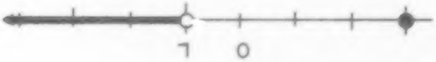
(c) $v \geq {}^-(\frac{3}{2})$.



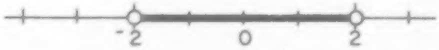
(d) $r \neq {}^-2$.



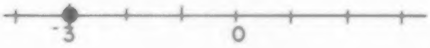
(e) $x = 3$ or $x < {}^-1$.



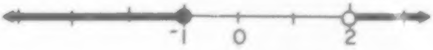
(f) $c < 2$ and $c > {}^-2$.



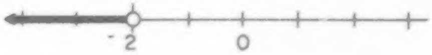
(g) $a \leq {}^-3$ and $a \geq {}^-3$.



(h) $d \leq {}^-1$ or $d > 2$.



(i) $a < 6$ and $a < {}^-2$.



(j) $u > 2$ and $u < {}^-3$. The truth set in this case is the empty set, $\emptyset$, and hence, has no graph. The reader should observe that there is no number both less than $^-3$ and greater than 2. The exercise may serve to remind the reader to notice carefully whether the connective in the sentence is "and" or "or", and to interpret the sentence accordingly.

3. (a) A sentence whose truth set is the set of all real numbers not equal to 3 is: $y \neq 3$. Another is: $y > 3$ or $y < 3$. Note that although the sentences above are mathematically equivalent, their English translations differ, and it is the former sentence which describes the required set directly.

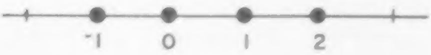
   (b) A sentence whose truth set is the set of all real numbers less than or equal to $^{-}2$ is $v \leq {}^{-}2$. Another is: $v \ngtr {}^{-}2$.

   (c) A sentence whose truth set is the set of all real numbers not less than $^{-}(\frac{5}{2})$ is $x \nless {}^{-}(\frac{5}{2})$. Another is: $x \geq {}^{-}(\frac{5}{2})$. Notice here that the alternate form is easier to comprehend. This may suggest to the reader a clearer description in English for the required set.

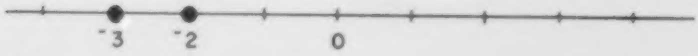
4. If p is any positive real number, and n is any negative real number, then n is to the left of zero, and p is to the right of zero; thus, n is to the left of p. (Recall that this principle was used to speed the comparison of numbers in many of the preceding exercises.) It follows that "$n < p$" and "$n \neq p$" are true statements and "$p < n$" is false. The statement "$n \leq p$" is true, since the statement means $n < p$ or $n = p$, and, though the second statement is false, the first is true.

5. If p is any number of the set of integers, the truth set of:

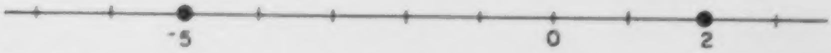
   (a) $^{-}2 < p$ and $p < 3$ is $\{^{-}1, 0, 1, 2\}$. In words, this is the set of all integers both greater than $^{-}2$ and less than 3. On the number line the set would have the graph:



(b) $p \leq {}^-2$ and ${}^-4 < p$, is $\{{}^-2, {}^-3\}$. This is the set of all integers both less than ${}^-4$ and greater than or equal to ${}^-2$. Its graph is:



(c) $p = 2$ or $p = {}^-5$ is $\{{}^-5, 2\}$. In words, we have the direct description of this set: the set of integers equal to 2 or ${}^-5$. Its graph is:



6. Use =, <, or > to relate each of the following pairs so as to make a true sentence.

(a) $\frac{3}{5} > {}^-(\frac{6}{10})$.

Notice that in this and in most of the following exercises the multiplication property of 1 may be used to facilitate the comparison.

(b) $\frac{3}{5} > \frac{3}{6}$.

(c) $\frac{9}{12} > \frac{8}{12}$.

(d) ${}^-(\frac{173}{29}) > {}^-(\frac{183}{29})$.

(e) ${}^-(\frac{3}{5}) < \frac{3}{6}$.

(f) ${}^-(\frac{3}{5}) < {}^-(\frac{3}{6})$.

Answers to Problems; page 80:

1. If a is a real number and b is a real number, then exactly one of the following is true:

2. The best statement would be: "If a is a real number and b is a real number, then exactly one of the following is true: $a \geq b$ or $a < b$.
Some may say "For any real numbers a and b, $a \geq b$ or $a \leq b$. If $a \leq b$ and $b \leq a$, then $a = b$". The last sentence of this particular statement is reasonable and innocent in appearance. Surprisingly enough, it turns out to be one of the most useful criteria for determining that two variables have the same value! In many instances in the calculus, for example, one is able to show by one argument that $a \leq b$ and by another that $b \leq a$. He is then able to conclude that $a = b$. Given two numerals, it is usually trivial to check whether or not they name the same number. In the case of two numbers, of course, we have complete information. It is only when our information about two "numerals" is incomplete that a statement like, "If $a \leq b$ and $b \leq a$, then $a = b$", can possibly be useful as a tool.

Answers to Problems; pages 81-82:

1. (a) $^-(\frac{1}{5}) < \frac{3}{2}$, $\frac{3}{2} < 12$, $^-(\frac{1}{5}) < 12$.

   (b) $^-\pi < {^-\sqrt{2}}$, $^-\sqrt{2} < \pi$, $^-\pi < \pi$.

   (c) $^-1.7 < 0$, $0 < 1.7$, $^-1.7 < 1.7$.

   (d) $^-(\frac{27}{15}) < {^-(\frac{3}{15})}$, $^-(\frac{3}{15}) < {^-(\frac{2}{15})}$, $^-(\frac{27}{15}) < {^-(\frac{2}{15})}$.

   (e) $^-(\frac{1}{2}) = {^-(\frac{6}{12})}$, $^-(\frac{1}{3}) = {^-(\frac{4}{12})}$, $^-(\frac{1}{4}) = {^-(\frac{3}{12})}$. Thus,
   $^-(\frac{6}{12}) < {^-(\frac{4}{12})}$, $^-(\frac{4}{12}) < {^-(\frac{3}{12})}$, $^-(\frac{6}{12}) < {^-(\frac{3}{12})}$, and
   $^-(\frac{1}{2}) < {^-(\frac{1}{3})}$, $^-(\frac{1}{3}) < {^-(\frac{1}{4})}$, $^-(\frac{1}{2}) < {^-(\frac{1}{4})}$.

(f) $1 + \frac{1}{2} = \frac{3}{2}$ or $\frac{6}{4}$.

$1 + (\frac{1}{2})^2 = 1 + \frac{1}{4} = \frac{5}{4}$.

$(1 + \frac{1}{2})^2 = (\frac{3}{2})^2 = \frac{9}{4}$, or

$\frac{5}{4} < \frac{6}{4}$, $\frac{6}{4} < \frac{9}{4}$, $\frac{5}{4} < \frac{9}{4}$, or

$1 + (\frac{1}{2})^2 < 1 + \frac{1}{2}$, $1 + \frac{1}{2} < (1 + \frac{1}{2})^2$, $1 + (\frac{1}{2})^2 < (1 + \frac{1}{2})^2$.

2. Of three real numbers a, b, and c, if $a > b$ and $b > c$, then $a > c$.

3. The transitive property for "=" is: For all real numbers a, b, and c, if $a = b$ and $b = c$, then $a = c$.

   If Art weighs the same as Bob and Bob and Cal are equal in weight, we know Art and Cal must weigh the same. If $3 + 4 = 7$ and $7 = 5 + 2$, then $3 + 4 = 5 + 2$.

4. The transitive property for $\geq$ would be: For all real numbers a, b and c, if $a \geq b$ and $b \geq c$, then $a \geq c$. If $\pi \geq 3.14$ and $3.14 \geq 2$, then $\pi \geq 2$.

5. (a) The non-positive real numbers are the set of numbers less than or equal to 0; in other words, the set comprised of 0 and all negative numbers.

   (b) The non-negative real numbers are the set of numbers greater than or equal to 0; in other words, the set consisting of zero and all the positive numbers.

6. (a) $^{-}(\frac{145}{28})$ and $^{-}(\frac{104}{21})$.

Using mixed numbers, $^{-}(\frac{145}{28}) = {}^{-}(5\frac{5}{28})$ and $^{-}(\frac{104}{21}) = {}^{-}(4\frac{20}{21})$. If $^{-}(5\frac{5}{28}) < {}^{-}5$ and $^{-}5 < {}^{-}(4\frac{20}{21})$, then $^{-}(5\frac{5}{28}) < {}^{-}(4\frac{20}{21})$ and $^{-}(\frac{145}{28}) < {}^{-}(\frac{104}{21})$. Alternatively, if $^{-}(\frac{145}{28}) < {}^{-}(\frac{140}{28})$ or $^{-}5$, and $^{-}(\frac{105}{21})$ or $^{-}5 < {}^{-}(\frac{104}{21})$, then $^{-}(\frac{145}{28}) < {}^{-}(\frac{104}{21})$.

(b) $^{-}(\frac{192}{46})$ and $^{-}(\frac{173}{44})$.

If $^{-}(\frac{192}{46}) < {}^{-}(\frac{184}{46})$ or $^{-}4$, and $^{-}(\frac{176}{44})$ or $^{-}4 < {}^{-}(\frac{173}{44})$, then $^{-}(\frac{192}{46}) < {}^{-}(\frac{173}{44})$.

Answers to Problems; page 85:

1. (a) The opposite of 2.3 is -2.3.

   (b) The opposite of -2.3 is 2.3.

   (c) The opposite of -(-2.3) is -2.3. Note here that the opposite of the opposite of a number is the number itself.

   (d) The opposite of -(3.6 - 2.4) is 3.6 - 2.4, or more simply, 1.2.

   (e) The opposite of -(42 × 0) is (42 × 0) or more simply, 0.

   (f) The opposite of -(42 + 0) is (42 + 0) or more simply 42.

   Exercises (e) and (f) provide an opportunity to see whether the zero properties for addition and multiplication have lodged in the readers' minds.

2. (a) If x is positive, then the opposite of x is negative.

   (b) If x is negative, then the opposite of x is positive.

   (c) If x is zero, then the opposite of x is zero.

3. (a) If the opposite of x is a positive number, then the number x itself must be negative.

   (b) If the opposite of x is a negative number, then the number itself must be positive.

   (c) If the opposite of x is 0, then the number x itself must be 0, for 0 is its own opposite.

4. (a) Since every real number has an opposite, it follows that every real number is the opposite of some real number.

   (b) Yes. See 4(a).

   (c) Every negative number is the opposite of some (positive) real number; hence, the set of negative numbers is a subset of the set of all opposites.

   (d) Some opposites, namely, opposites of negative numbers, are not negative numbers. Hence, the set of opposites is not a subset of the set of negative numbers.

   (e) No. See 4(d).

Answers to Problems; pages 86-87:

1. (a) $-\frac{1}{6} < \frac{2}{7}$ and $-\frac{2}{7} < \frac{1}{6}$.

   (b) $-\pi < \sqrt{2}$ and $^{-}\sqrt{2} < \pi$.

   (c) $\pi < \frac{22}{7}$ and $-\frac{22}{7} < -\pi$.

   The students may need to be told that $\pi = 3.1416$ to 4 decimal places and they may determine $\frac{22}{7} = 3.1428$ to four decimal places.

   (d) $3(\frac{4}{3} + 2) = 3 \times \frac{4}{3} + 3 \times 2$ by the Distributive property
   $= 4 + 6$
   $= 10.$

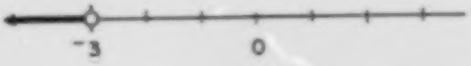
   $\frac{5}{4}(20 + 8) = (\frac{5}{4})(20) + (\frac{5}{4})8$ by the Distributive Property
   $= 25 + 10$
   $= 35.$

   Since $10 < 35$ and $-35 < -10$, we have
   $3(\frac{4}{3} + 2) < \frac{5}{4}(20 + 8)$ and $-\frac{5}{4}(20 + 8) < -3(\frac{3}{4} + 2)$.

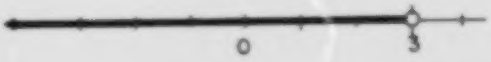
   (e) $-(\frac{8 + 6}{7}) = -\frac{14}{7} = -2.$

   Then $-(\frac{8 + 6}{7})$ and -2 are names for the same number.

2. (a) $-x > 3.$



   (b) $-x > -3.$



3. (a) The sentence states "the opposite of x is not equal to 3". There are many numbers whose opposites are not equal to 3; in fact, there is only one number whose opposite does equal 3, and this is, of course, -3. Hence, the required set is the set of all real numbers except -3.

   (b) $-x \neq -3$.
   By the reasoning of Part (a) the required set here is found to be the set of all real numbers except 3.

   (c) $-x < 0$.
   Here the set required is the set of all real numbers whose opposites are less than zero. Now if the opposites of all members of this set are less than zero, the members of the set must be greater than zero; in other words, "$-x < 0$" and "$x > 0$" have the same truth set. Thus, the truth set is the set of all positive real numbers.

   (d) $-x \leq 0$.
   Here the reasoning would parallel (c) above: Each member of the truth set is either zero or a positive number. This set is described briefly as the non-negative numbers.

4. (a) $x < 1$.

   (b) $-2 < x$ and $x \leq 1$.
   This open sentence can be written much more suggestively as:
   $-2 < x \leq 1$.
   We would read this "x is greater than -2 and less than or equal to 1". This terminology emphasizes the number line picture and suggests strongly that x is 1 or is between -2 and 1.

We never write, for example, "$-2 < x \geq 1$" as a shorthand for "$-2 < x$ or $x \geq 1$", but read "$-2 < x \geq 1$" as "x is greater than -2 and greater than or equal to 1"; in other words, we would read "$-2 < x \geq 1$" as a conjunction, when what is wanted is a disjunction.

(c) $x \leq -1$ or $x > 1$.

(d) $x > -2$ and $x < 2$ or, more briefly, $-2 < x < 2$.

5. (a) -3 is the opposite of 3. The greater is 3.

(b) -0 is the opposite of 0. They are the same number.

(c) $\sqrt{2}$ is the opposite of $-\sqrt{2}$. The greater is $\sqrt{2}$.

(d) -17 is the opposite of 17. The greater is 17.

(e) 0.01 is the opposite of -0.01. The greater is 0.01.

(f) -2 is the opposite of $-(-2)$. The greater is $-(-2)$.

(g) $(\frac{1}{2} - \frac{1}{3})$ is the opposite of $-(\frac{1}{2} - \frac{1}{3})$. The greater is $(\frac{1}{2} - \frac{1}{3})$.

The opposites of (c), (e), (f) and (g) may be given in terms of the opposites of the opposites, e.g.,
(c) The opposite of $\sqrt{2}$ is $-(-\sqrt{2})$, etc.

6. The relation "$\}$" does not have the comparison property. For example, 2 and -2 are different real numbers but neither is further from 0 than the other; in other words, none of the statements "$-2 = 2$", "$-2 \} 2$" and "$2 \{ -2$" is true.

The transitive property for "$\}$" would read: If a, b, and c, are real numbers and if $a \} b$ and $b \} c$, then $a \} c$. This is certainly a true statement as can be seen by substituting the phrase "is further from 0 than" for "$\{$" wherever it occurs.

The relations "≻" and ">" have the same meaning for the numbers of arithmetic: "is further from 0 than" and "is to the right of" mean the same thing on the arithmetic number line.

7. Following the hint:

$-\frac{13}{42}$ and $-\frac{15}{49}$ are to be compared.

$\frac{13}{42}(\frac{7}{7}) = \frac{91}{294}$ (Multiplication property of 1.)

$\frac{15}{49}(\frac{6}{6}) = \frac{90}{294}$

$\frac{91}{294} > \frac{90}{294}$ and $-\frac{91}{294} < -\frac{90}{294}$. Thus, $-\frac{13}{42} < -\frac{15}{49}$.

In order to compare two negative rational numbers, we use the multiplication property of 1 to compare their opposites and then use the property of opposites: For real numbers a and b, if $a < b$, then $-b < -a$. We can describe this briefly as: In order to compare two negative (or two positive) rational numbers, represent them by fractions with the same denominator and compare the numbers represented by their numerators.

Answers to Problems; page 89:

1. (a) 7; the greater of -7 and 7 is 7.

   (b) 3; the greater of $-(-3)$ and its opposite -3 is $-(-3)$ or 3.

   (c) 2; $6 - 4$ is another name for 2, and 2 is greater than its opposite, -2.

   (d) 0; by the multiplication property of 0, (14×0) is 0, and the absolute value of 0 is 0.

(e) 14; by the addition property of 0, $(14 + 0)$ is 14, and 14 is greater than its opposite, -14.

(f) 3; the opposite of the opposite of the opposite of 3 is simply -3 and the opposite of -3, 3, is greater than -3.

2. (a) Both are non-negative.

(b) If x is a non-negative number, it corresponds to a point at or to the right of 0 on the real number line. Its opposite, then, is at or to the left of 0. It follows that the greater of x and its opposite $-x$ is here x. By definition, then, $|x|$ is x, a non-negative number.

3. (a) and (b)

If x is a negative number, it corresponds to a point to the left of 0 on the real number line. Its opposite is therefore to the right of 0. Thus, the greater of x and $-x$ is, in this case, $-x$; in other words, if x is a negative number $|x|$ is $-x$, the opposite of x, and thus a positive number.

4. For every real number x, $|x|$ is a non-negative number. In Parts (a) and (b) all cases, $x < 0$, $x = 0$, $x > 0$, have been considered, and in every case, $|x|$ was found to be non-negative.

5. For the negative number x, $|x|$ is greater than x since, for x negative, $|x|$ is positive by Problem 2(b). Since any negative number is less than any positive number, $x < |x|$ for all negative x.

6. The set $\{-1, -2, 1, 2\}$ is closed under the operation of taking the absolute value of its elements. Taking the absolute value of each element of the set,

$$|-1| = 1$$
$$|-2| = 2$$
$$|1| = 1$$
$$|2| = 2,$$

we find that the set of absolute values of the numbers of the original set to be $\{1, 2\}$. Since, $\{1, 2\}$ is a subset of $\{-2, -1, 1, 2\}$, this latter set is closed under the operation of taking absolute values of its elements.

Answers to Problems; pages 89-91:

1. (a) $|-7| < 3$ or $7 < 3$. False
   (b) $|-2| \leq |-3|$ or $2 \leq 3$. True
   (c) $|4| < |1|$ or $4 < 1$. False
   (d) $2 \not< |-3|$ or $2 \not< 3$. False
   (e) $|-5| \not< |2|$ or $5 \not< 2$. True
   (f) $-3 < 17$. True
   (g) $-2 < |-3|$ or $2 < 3$. True
   (h) $|\sqrt{16}| > |-4|$ or $4 > 4$. False
   (i) $|-2|^2 = 4$ or $2^2 = 4$. True
   (b). (e), (f), (g) and (i) are true.

2. (a) $|2| + |3| = 2 + 3 = 5$.
   (b) $|-2| + |3| = 2 + 3 = 5$.
   (c) $-(|2| + |3|) = -(2 + 3) = -5$.
   (d) $-|-2| + |3| = -2 + 3 = 1$.
   (e) $|-7| - (7 - 5) = 7 - 2 = 5$.
   (f) $7 - |-3| = 7 - 3 = 4$.
   (g) $|-5| \times 2 = 5 \times 2 = 10$.
   (h) $-(|-5| - 2) = -(5 - 2) = -3$.
   (i) $|-3| - |2| = 3 - 2 = 1$.
   (j) $|-2| + |-3| = 2 + 3 = 5$.

(k) $-(|-3| - 2) = -(3 - 2) = -1$.

(l) $-(|-2| + |-3|) = -(2 + 3) = -5$.

(m) $3 - |3 - 2| = 3 - 1 = 2$.

(n) $-(|-7| - 6) = -(7 - 6) = -1$.

(o) $|-5| \times |-2| = 5 \times 2 = 10$.

(p) $-(|-2| \times 5) = -(2 \times 5) = -10$.

(q) $-(|-5| \times |-2|) = -(5 \times 2) = -10$.

3. (a) $|x| = 1$. The truth set is $\{1, -1\}$.

(b) $|x| = 3$. The truth set is $\{3, -3\}$.

(c) $|x| + 1 = 4$. The truth set is $\{3, -3\}$, the same as that of $|x| = 3$.

(d) $5 - |x| = 2$. The truth set is $\{3, -3\}$.

4. (a) $|x| \geq 0$ is true for all real numbers x.
If $x \geq 0$, $|x| \geq 0$. See Problem 2(b) on page 89.
If $x < 0$, $|x| > 0$. See Problem 3(b) on page 89.

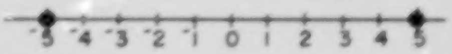
(b) $x \leq |x|$ is true for all real numbers x.
If $x \geq 0$, $x = |x|$. If $x < 0$, $x < |x|$.

(c) $-x \leq |x|$ is true for all real numbers x.
If $x \geq 0$, $-x \leq |x|$. If $x < 0$, $-x = |x|$.

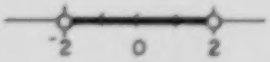
(d) $-|x| \leq x$ is true for all real numbers x.
If $x \geq 0$, $-|x| \leq x$. If $x < 0$, $-|x| = x$.

5. Graph the truth sets of the following sentences:

(a) $|x| = 5$.

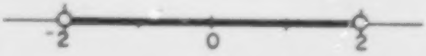


(b) $|x| < 2$.

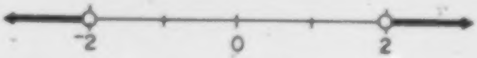


The reader may arrive at the required graph by trial of different numbers for x in the sentence. He may instead reason the exercise out somewhat as follows: The sentence states that "The absolute value of x is less than 2". On the number line, this statement becomes "x is less than 2 units away from 0". Therefore, the graph of "$|x| < 2$" is the one given on the previous page.

(c) $x > -2$ and $x < 2$.

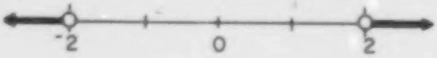


(d) $|x| > 2$.



As in Part (b) the reader here may find the required set by trial-and-error, or by recalling the interpretation of absolute value as a distance on the number line as in (b) on the previous page.

(e) $x < -2$ or $x > 2$.



(f) Since the absolute of a number is always a non-negative number this sentence is meaningless.

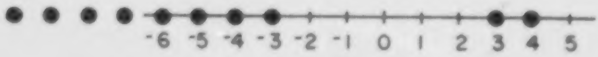
6. The set of integers less than 5 is the set

$$\{\ldots, -1, 0, 1, 2, 3, 4\}.$$

The set of integers less than 5 whose absolute values are greater than 2 is

$$\{\ldots, -5, -4, -3, 3, 4\}.$$

-5 and -10 are both elements of this set, but 0 is not.



7. Three numbers:

(a) In P but not in I: All positive numbers except integers, $\frac{3}{10}$, $\sqrt{2}$, π, 5.3, etc.

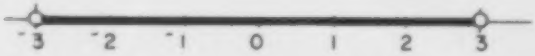
(b) In R but not in P: All non-positive real numbers, -7, $-\pi$, 0, $-\sqrt{2}$, $-\frac{5}{3}$, etc.

(c) In R but not in P or I: All non-positive real numbers, except integers, $-\frac{17}{5}$, -2.74, $-\frac{\pi}{2}$, $-\sqrt{2}$, etc.

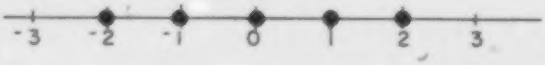
(d) In P but not in R: All non-real positive numbers. Since there are none, this is the empty set, $\emptyset$.

Answers to Review Problems; page 92:

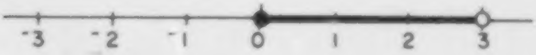
1. (a)



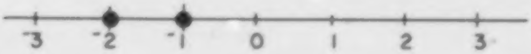
(b)



(c)

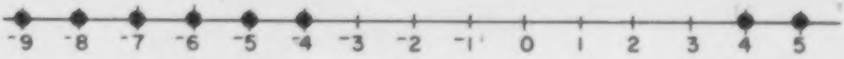


(d)

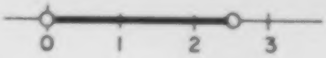


2. (a), (b), and (c) are true statements.

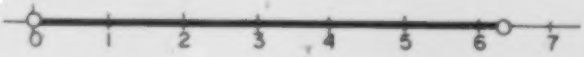
3. etc.


-9
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5

4. (a) If s is the number of units in the side of this square, s is positive and $4s$ is the perimeter of the square. A sentence for this is $s > 0$ and $4s < 10$.

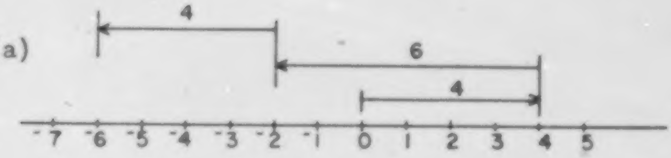


(b) If A is the number of units in the area of the square, then $A = s^2$, where $s > 0$ and $4s < 10$ as in Part (a). Since A is s^2, and s is a number from the set of numbers between 0 and 2.5, the truth set of A is the set of numbers between 0 and 6.25.



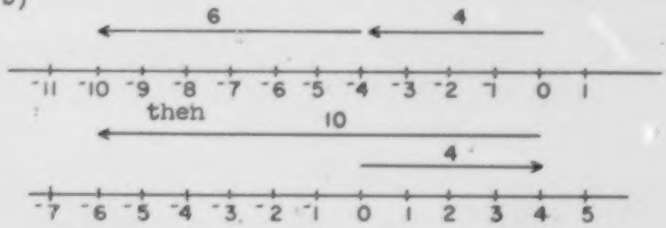
Answers to Problems; page 94:

1. (a)


4
6
4
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5

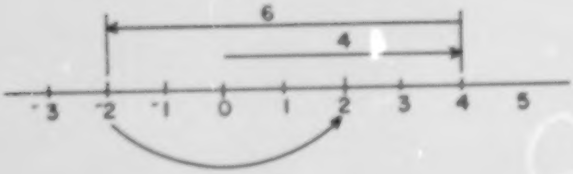
The sum is -6.

(b)


6
4
-11
-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
1
then
10
4
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5

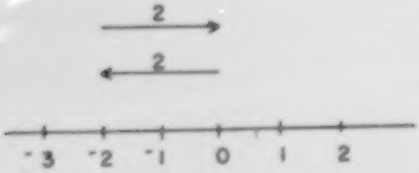
The sum is -6.

(c)

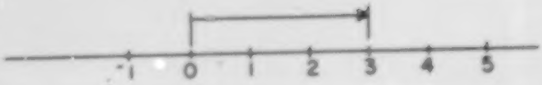

6
4
-3
-2
-1
0
1
2
3
4
5

The sum is 2.

(d)

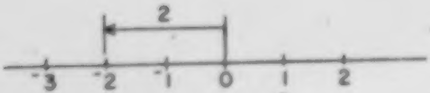


then

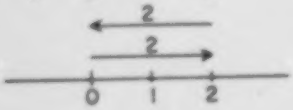


The sum is 3.

(e)

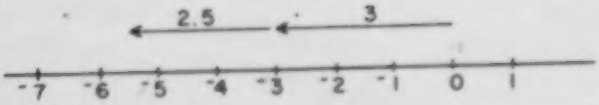


then



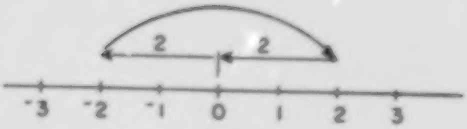
The sum is 0.

(f)


2.5
3
-7
-6
-5
-4
-3
-2
-1
0
1

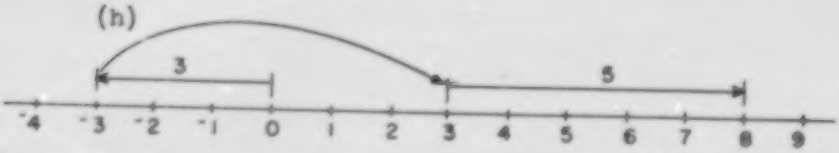
The sum is -5.5.

(g)

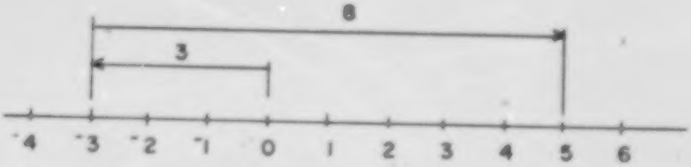


The sum is 0.

(h)



then



The sum is 5.

2. (a) Move from 0 to 7 on the number line, then move 10 units to the right.

(b) Move from 0 to 7 on the number line, then move 10 units to the left.

(c) Move from 0 to 10 on the number line, then move 7 units to the left.

(d) Move from 0 to -10 on the number line, then move 7 units to the left.

(e) Move from 0 to 10 on the number line, then move 7 units to the right.

(f) Move from 0 to -7 on the number line, then move 10 units to the left.

(g) Move from 0 to -7 on the number line, then move 10 units to the right.

(h) Move from 0 to -10 on the number line, then move 7 units to the right.

(i) Move from 0 to -10 on the number line, then move 0 units.

(j) Move from 0 to 0 on the number line, then move 7 units to the right.

3. In (a), (e) and (j).

4. When both numbers were negative, the sum was negative, and was the opposite of the sum obtained when both numbers were positive.

Answers to Problems; pages 97-98:

1. (a) $(-2) + (-7) = -(|-2| + |-7|)$

$= -(2 + 7)$

$= -9$

A loss of $2 followed by a loss of $7 is a net loss of $9.

(b) $(-4.6) + (-1.6) = -(|-4.6| + |-1.6|)$

$= -(4.6 + 1.6)$

$= -6.2$

Move from 0 to -4.6 on the number line, then move 1.6 units to the left. You arrive at -6.2.

(c) $(-3\frac{1}{3}) + (-2\frac{2}{3}) = -(|-3\frac{1}{3}| + |-2\frac{2}{3}|)$

$= -(3\frac{1}{3} + 2\frac{2}{3})$

$= -6$

Move from 0 to $-3\frac{1}{3}$ on the number line, then move $2\frac{2}{3}$ units to the left. You arrive at -6.

(d) $(-25) + (-73) = -(|-25| + |-73|)$
$= -(25 + 73)$
$= -98$

A loss of \$25 followed by a loss of \$73 is a net loss of \$98.

(e) $5\frac{1}{2} + 2\frac{1}{2} = 8$

Here we have a problem involving only the addition of positive numbers, so that the definition for the addition of negative numbers cannot be used.

2. (a) $(-6) + (-7) = -13$

(b) $(-7) + (-6) = -13$

(c) $-(|-7| + |-6| = -(7 + 6)$
$= -13$

(d) $6 + (-4) = 2$

(e) $(-4) + 6 = 2$

(f) $|6| - |-4| = 6 - 4$
$= 2$

(g) $0 + (-3) = -3$

(h) $-(|-3| - |0|) = -(3 - 0)$
$= -3$

(i) $3 + \left((-2) + 2\right) = 3 + 0$
$= 3$

3. From the point of view of the number line: If the distance moved to the right was greater than the distance moved to the left, the sum was positive; if the distance moved to the left was greater, the sum was negative.

In terms of absolute value: If the negative number has the greater absolute value, the sum is negative; if the positive number has the greater absolute value, the sum is positive.

Answers to Problems; pages 99-101:

1. (a) $(-5) + 3 = -(|5| - |3|)$
$= -(5 - 3)$
$= -2$

(b) $(-11) + (-5) = -(|-11| + |-5|)$
$= -(11 + 5)$
$= -16$

(c) $(-\frac{8}{3}) + 0 = -(|-\frac{8}{3}| - |0|)$
$= -(\frac{8}{3} - 0)$
$= -\frac{8}{3}$

(d) $2 + (-2) = |2| - |-2|$
$= 2 - 2$
$= 0$

(e) $18 + (-14) = |18| - |-14|$
$= 18 - 14$
$= 4$

(f) $12 + 7.4 = 19.4$

(g) $(-\frac{2}{3}) + 5 = |5| - |-\frac{2}{3}|$
$= 5 - \frac{2}{3}$
$= \frac{15}{3} - \frac{2}{3}$
$= \frac{13}{3}$

(h) $(-35) + (-65) = -(|-35| + |-65|)$
$= -(35 + 65)$
$= -100$

2. Since the sum of two real numbers is a real number, the set of all real numbers is closed under addition.

3. Since the sum of two negative real numbers is a negative real number, the set of all negative real numbers is closed under addition.

4. (a) If x is 5, then $5 + 2 = 7$ is true.

   (b) If y is -10, then $3 + (-10) = -7$ is true.

   (c) If a is -5, then $(-5) + 5 = 0$ is true.

   (d) If b is 10, then $10 + (-7) = 3$ is true.

   (e) If x is 0, then $(-\frac{5}{6}) + 0 = -\frac{5}{6}$ is true.

   (f) If c is -4, then $(-4) + (-3) = -7$ is true.

   (g) If y is $-\frac{9}{6}$, then $(-\frac{9}{6}) + \frac{2}{3} = -\frac{5}{6}$ is true.

   (h) If x is 20 then $\frac{1}{2}(20) + (-4) = 6$ is true.

   (i) If y is 3, then $\left(3 + (-2)\right) + 2 = 3$ is true.

   (j) If x is (-1), then $\left(3 + (-1)\right) + (-3) = -1$ is true.

5. (a) True

   (b) True

   (c) True

   (d) False

   (e) False

6. (a) If x is the distance from the starting point (with north taken as the positive direction), then

   $$x = 40 + (-55).$$

   (b) If n is the third number, then

   $$(-9) + 28 + n = (-52).$$

   (c) If c is the temperature change between 4 P.M. and 8 P.M., then

   $$-2 + 15 + 6 + c = -9.$$

   (d) If g is the number of pounds gained the third week, then

   $$200 + (-4) + (-6) + g = 195.$$

   (e) If s is the number of points change in the stock price listing, then

   $$83 + (-5) + s = 86.$$

Answers to Problems; pages 103-104:

1. (a) The left numeral is

$\left(3 + (-3) + 4\right) = \left(3 + (-3)\right) + 4$ associative property of addition.

$= 0 + 4.$ addition property of opposites.

The right numeral is

$0 + 4.$

(b) The right numeral is

$\left((-3) + 5\right) + 7 = \left(5 + (-3)\right) + 7$ commutative property of addition.

The left numeral is

$\left(5 + (-3)\right) + 7.$

(c) The left numeral is

$\left(7 + (-7)\right) + 6 = 0 + 6$ addition property of opposites.

$= 6$ addition property of 0.

The right numeral is 6.

(d) The left numeral is

$|-1| + |-3| + (-3) = 1 + 3 + (-3)$ definition of absolute value

$= 1 + \left(3 + (-3)\right)$ associative property of addition.

$= 1 + 0.$ addition property of opposites.

$= 1$ addition property of 0.

The right numeral is 1.

(e) The right numeral is

$\left((-2) + 3\right) + (-4) = (-2) + \left(3 + (-4)\right)$ associative property of addition.

The left numeral is

$(-2) + \left(3 + (-4)\right).$

(f) The left numeral is

$(-|-5|) + 6 = (-5) + 6$ — definition of absolute value.

$ = 6 + (-5).$ — commutative property of addition.

The right numeral is

$6 + (-5).$

2. (a)

$x = x + (-x) + 3$

$x = \big(x + (-x)\big) + 3$

$x = 0 + 3$

$x = 3$

(b)

$m + 7 + (-m) = m$

$m + (-m) + 7 = m$

$\big(m + (-m)\big) + 7 = m$

$0 + 7 = m$

$7 = m$

(c)

$n + (n + 2) + (-n) + 1 + (-3) = 0$

$n + (-n) + (n + 2) + 1 + (-3) = 0$

$\big(n + (-n)\big) + n + (2 + 1) + (-3) = 0$

$0 + n + \big(3 + (-3)\big) = 0$

$n + 0 = 0$

$n = 0$

(d)

$(y + 4) + (-4) = 9 + (-4)$

$y + \big(4 + (-4)\big) = 9 + (-4)$

$y + 0 = 9 + (-4)$

$y = 9 + (-4)$

$y = 5$

Answers to Problems; page 107:

1. If $x + 5 = 13$ is true for some x, then
   $x + 5 + (-5) = 13 + (-5)$ is true for the same x,
   $x + 0 = 8$ is true for the same x,
   $x = 8$ is true for the same x.

   If $x = 8$,
   the left member is $8 + 5 = 13$;
   the right member is 13.
   Hence, the truth set is $\{8\}$;
   the only solution is 8.

2. If $(-6) + 7 = (-8) + x$ is true for some x, then
   $8 + (-6) + 7 = 8 + (-8) + x$ is true for the same x,
   $9 = 0 + x$ is true for the same x,
   $9 = x$ is true for the same x.

   If $x = 9$,
   the left member is $(-6) + 7 = 1$,
   the right member is $(-8) + 9 = 1$.
   Hence, the truth set is $\{9\}$.

3. If $(-1) + 2 + (-3) = 4 + x + (-5)$ is true for some x,
   then $(-2) = x + (-1)$ is true for the same x,
   $(-2) + 1 = x + (-1) + 1$ is true for the same x,
   $-1 = x$ is true for the same x.

   If $x = -1$,
   the left member is $(-1) + 2 + (-3) = (-2)$,
   the right member is $4 + (-1) + (-5) = (-2)$.
   The solution is -1.

4. If $(x + 2) + x = (-3) + x$ is true for some x, then
   $2x + 2 = (-3) + x$ is true for the same x,
   $2x + 2 + (-x) + (-2)$
   $= (-3) + x + (-x) + (-2)$ is true for the same x,
   $x = -5$ is true for the same x.

If $x = -5$,

the left member is $\big((-5) + 2\big) + 5 = (-3) + 5$

$= 2$,

the right member is $(-3) + 5 = 2$.

Hence, the truth set is $\{-5\}$.

5. If $(-2) + x + (-3) = x + (-\frac{5}{2})$ is true for some x,

then $x + (-5) = x + (-\frac{5}{2})$ is true for the same x,

$(-x) + x + (-5)$

$= (-x) + x + (-\frac{5}{2})$ is true for the same x,

$-5 = -\frac{5}{2}$ is true for the same x;

but $-5 = -\frac{5}{2}$ is false, which contradicts the assumption that the equation was true for some x.

Hence, the truth set is $\emptyset$.

6. If $|x| + (-3) = |-2| + 5$ is true for some x, then

$|x| + (-3) + 3 = 2 + 5 + 3$ is true for the same x,

$|x| = 10$ is true for the same x,

$x = 10$ or $x = -10$ is true for the same x.

If $x = 10$,

the left member is $|10| + (-3) = 10 + (-3)$

$= 7$,

the right member is $|-2| + 5 = 2 + 5$

$= 7$.

If $x = -10$,

the left member is $|-10| + (-3) = 10 + (-3)$

$= 7$,

the right member is $|-2| + 5 = 2 + 5$

$= 7$.

The solutions are 10, -10.

7. If $(-\frac{3}{8}) + |x| = (-\frac{3}{4}) + (-1)$ is true for some x, then

$\frac{3}{8} + (-\frac{3}{8}) + |x| = \frac{3}{8} + (-\frac{3}{4}) + (-1)$ is true for the same x,

$|x| = \frac{3}{8} + (-\frac{6}{8}) + (-\frac{8}{8})$ is true for the same x,

$|x| = -\frac{11}{8}$ is true for the same x.

But $|x|$ is non-negative for every x, which contradicts the assumption that the equation is true for some x. There is no solution.

8. If $x + (-3) = |-4| + (-3)$ is true for some x, then

$x + (-3) + 3 = 4 + (-3) + 3$ is true for the same x,

$x = 4$ is true for the same x.

If $x = 4$,

the left member is $4 + (-3) = 1$,

the right member is $|-4| + (-3) = 4 + (-3)$,

$= 1$.

Hence, the truth set is $\{4\}$.

9. If $(-\frac{4}{3}) + (x + \frac{1}{2}) = x + (x + \frac{1}{2})$ is true for some x, then

$x + (-\frac{8}{6}) + \frac{3}{6} = 2x + \frac{3}{6}$ is true for the same x,

$x + (-x) + (-\frac{3}{6}) + (-\frac{5}{6}) = 2x + (-x) + (-\frac{3}{6}) + \frac{3}{6}$ is true for the same x.

$-\frac{8}{6} = x$

$-\frac{4}{3} = x$

If $x = -\frac{4}{3}$

the left member is $(-\frac{4}{3}) + (-\frac{4}{3} + \frac{1}{2}) = (-\frac{8}{6}) + (-\frac{8}{6}) + \frac{3}{6}$

$= -\frac{13}{6}$,

the right member is $-\frac{4}{3} + (-\frac{4}{3} + \frac{1}{2}) = -\frac{8}{6} + (-\frac{8}{6}) + \frac{3}{6}$

$= -\frac{13}{6}$.

The solution is $-\frac{4}{3}$.

Answers to Problems; page 110:

1. In each part of this problem, the number for which the sentence is true is determined almost immediately by the uniqueness of the additive inverse; i.e. if $x + z = 0$, then $z = -x$.

   (a) -3

   (b) 2

   (c) -8

   (d) $\frac{1}{2}$

   (e) -3

   (f) $-\frac{2}{3}$

   (g) $\frac{7}{3}$

   (h) 3

   (i) 3

2. Yes, for by this theorem there is but one possible value for the variable, the opposite of the number to which the variable is added to make 0.

Answers to Problems; pages 112-113:

1. (a) $-(x + y) = (-x) + (-y)$. True.

   (b) $-x = -(-x)$. Since $-(-x) = x$, this is false for all real x except zero.

   (c) $-(-x) = x$. True for all real x.

   (d) $-\left(x + (-2)\right) = (-x) + 2$. True; a special case of (a) where $y = -2$, $-y = 2$.

   (e) $-\left(a + (-b)\right) = (-a) + b$. True; a special case of (a) where $x = a$, $y = -b$, $(-x) = (-a)$, $(-y) = b$.

   (f) For $a = 2$, $b = 4$, the sentence becomes

   $$\left(2 + (4)\right) + (-2) = 4$$
   $$-4 = 4$$

   which is false. The sentence is not true for all real numbers.

While this proof by counter-example is sufficient, one may also reason as follows: Application of the associative and commutative properties of addition, the addition property of opposites, and the addition property of 0 leads to $-b = b$; in other words, If "$\big(a + (-b)\big) + (-a) = b$" is true for all real numbers a and b, then "$-b = b$" is true for all real numbers a and b. But "$-b = b$" is true only for $b = 0$. Therefore, the statement "$\big(a + (-b)\big) + (-a) = b$" is not true for all real numbers a and b.

(g) $\big(-x + (-x)\big) = x + (-x)$. True; a special case of (a) where $x = x$, $y = (-x)$, $(-y) = x$.

2.

$(-x) + \big(y + (-z)\big) = (-x) + \big((-z) + y\big)$	commutative property of addition.
$= \big((-x) + (-z)\big) + y$	associative property of addition.
$= \big(-(x + z)\big) + y$	$-(a + b) = (-a) + (-b)$.
$= y + \big(-(x + z)\big)$	commutative property of addition.

3. $-\big(3 + 6 + (-4) + 5\big) = (-3) + (-6) + 4 + (-5)$ is true. The opposite of the sum of any number of numbers is the sum of their opposites.

(a) True

(b) False

(c) True

(d) False

4. For any real number a and any real number b and any real number c,

If	$a + c = b + c$,	
then	$(a + c) + (-c) = (b + c) + (-c)$	addition property of equality.
	$a + \big(c + (-c)\big) = b + \big(c + (-c)\big)$	associative property of addition.
	$a + 0 = b + 0$	addition property of opposites.
	$a = b$	addition property of 0.

Answers to Review Problems; pages 114-116:

1. (a) The left numeral is

$\frac{2}{3} + \left(7 + (-\frac{2}{3})\right) = \frac{2}{3} + \left((-\frac{2}{3}) + 7\right)$ commutative property for addition.

$= \left(\frac{2}{3} + (-\frac{2}{3})\right) + 7$ associative property for addition.

$= 0 + 7$ addition property of opposites.

$= 7.$ addition property of zero.

The right numeral is 7.

Hence, the sentence is true.

(b) The left numeral is

$|-5| + (-.36) + |-.36| = 5 + \left((-.36) + |.36|\right)$ associative property for addition.

$= 5 + \left((-.36) + (.36)\right)$ definition of absolute value.

$= 5 + 0$ addition property of opposites.

$= 5.$ addition property of zero.

The right numeral is $10 + ((2 + (-7)) = 10 + (-5)$

$= 5.$

Hence, the sentence is true.

2. (a) If $\frac{5}{9} + 32 = x + \frac{5}{9}$ is true for some x, then

$\frac{5}{9} + 32 + (-\frac{5}{9}) = x + \frac{5}{9} + (-\frac{5}{9})$ is true for the same x,

$\left(\frac{5}{9} + (-\frac{5}{9})\right) + 32 = x + 0$ is true for the same x,

$0 + 32 = x$ is true for the same x,

$32 = x.$ is true for the same x.

If $x = 32$,

the left member is $\frac{5}{9} + 32 = 32\frac{5}{9}$,

the right member is $32 + \frac{5}{9} = 32\frac{5}{9}$.

Hence, the truth set is $\{32\}$.

(b) If $x + 5 + (-x) = 12 + (-x) + (-3)$ is true for some x, then

$x + 5 + (-x) + x = 9 + (-x) + x$ is true for the same x,

$x + 5 = 9$ is true for the same x,

$x + 5 + (-5) = 9 + (-5)$ is true for the same x,

$x = 4.$ is true for the same x.

If $x = 4$,
the left member is $4 + 5 + (-4) = 5$,
the right member is $12 + (-4) + (-3) = 5$.
Hence, the truth set is $\{4\}$.

(c) If $3x + \frac{15}{2} + x = 10 + 3x + (-\frac{7}{2})$ is true for some x,

then $3x + \frac{15}{2} + x + (-3x) = 10 + 3x + (-\frac{7}{2}) + (-3x)$ is true for the same x,

$\frac{15}{2} + x = 10 + (-\frac{7}{2})$ is true for the same x,

$\frac{15}{2} + x + (-\frac{15}{2}) = 10 + (-\frac{7}{2}) + (-\frac{15}{2})$ is true for the same x,

$x = 10 - 11$ is true for the same x,

$x = -1.$ is true for the same x.

If $x = -1$,
the left member is $3(-1) + \frac{15}{2} + (-1) = (-3) + \frac{15}{2} + (-1)$

$= (-4) + \frac{15}{2}$

$= \frac{7}{2}$,

the right member is $10 + 3(-1) + (-\frac{7}{2}) = 10 + (-3) + (-\frac{7}{2})$

$= 7 + (-\frac{7}{2})$

$= \frac{7}{2}$.

Hence, the truth set is $\{-1\}$.

(d) If $|x| + 3 = 5 + |x|$ is true for some x,

then $|x| + 3 + (-|x|) = 5 + |x| + (-|x|)$ is true for the same x,

$3 = 5.$ is true for the same x.

But $3 = 5$ is false, which contradicts the assumption that the equation was true for some x. Hence, the truth set is $\emptyset$.

3. (a) $|3| + |a| > |-3|$ is true for the set of all numbers except 0.

(b) $|3| + |a| = |-3|$ is true for $\{0\}$.

(c) $|3| + |a| > |-3|$ is true for $\emptyset$.

4. (a) Either both are negative; or one is negative and the other is either positive or 0, and the negative number has the greater absolute value.

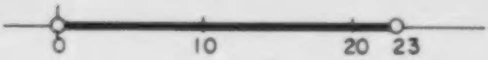
(b) One is the opposite of the other.

(c) Either both are positive; or one is positive and the other is either negative or 0, and the positive number has the greater absolute value.

5. (a) If x is the length of the fourth side,

$x > 0$ and $x < 23$.

(b)



6. (a) $2ab + ac$

(b) $2ab + 2ac$

(c) $3(a + b)$

(d) $5x(1 + 2a)$

(e) $xy(x + 1)$

(f) $2ab(3a + b)$

(g) $a^2bc + 3ab^2$

(h) $3a^2 + 6ab + 9ac$

7. (a) Yes, the set is closed under the operation of "opposite".

   (b) Yes, the set is closed under the operation of "absolute value".

   (c) Yes, if a set is closed under "opposite" it is closed under "absolute value", since either the number or its opposite is the absolute value of the number.

8. (a) Yes, the set is closed under the operation of "absolute value".

   (b) No, the set is not closed under the operation of "opposite".

   (c) No, if a set is closed under "absolute value", it is not necessarily closed under "opposite", since even if the absolute value of a number is in the set, the opposite of the number may not be.

Answers to Problems; pages 120-122:

1. (a) 5
(b) 5
(c) -9
(d) -9
(e) 19
(f) -5
(g) 0
(h) -12
(i) 12
(j) 1.35

2. (a) 36
(b) 36
(c) 4
(d) -35

3. (a) True
(b) True
(c) True

4. (a) {-3}
(b) {12}
(c) {36}
(d) all > -2
(e) $\{\frac{1}{3}, -\frac{1}{3}\}$

5. {φ}, closed

6. {1, 4, 6, 9, 16}

8. (a) Either both are positive or both negative
(b) One is positive and one is negative
(c) b is positive
(d) b is negative
(e) b is negative
(f) b is negative

9. 1. Comparison property
2. Multiplication property of 0
3. Definition of multiplication
4. Definition of opposites

Answers to Problems; page 124:

1. (b) -15 (d) 12

(c) 0 (e) -5

Answers to Problems; page 126:

2. (a) 5100 (c) 570

(b) 1 (d) -2100

Answers to Problems; page 126:

1. 900 3. $\frac{35}{12}$

2. -3 4. 75

Answers to Problems; page 127:

3. (a) $-5ab$ (c) $-21xy$

(b) $10ac$ (d) $-0.6cd$

Answers to Problems; page 128:

1. (a) $3x + 15$ (d) $(-y) + z + (-5)$

(b) $2a + 2b + 2c$ (e) $13y + xy$

(c) $3p + (-3q)$ (f) $(-gr) + (-g) + gs + gt$

2. (a) $5(a + b)$ (e) $(a + b)(x + y)$

(b) $(-9)(b + c)$ (f) $10(\frac{1}{5})$

(c) $6(5)$ (g) $(-6)(a^2 + b^2)$

(d) $3(x + y + z)$ (h) $c(a + b + 1)$

3. (a) $19t$
   (b) $-6a$
   (c) $13z$
   (d) $-11m$
   (e) $2a$
   (f) $4.0b$
   (g) $3a + 7y$
   (h) $16p$
   (i) $2a + 19b$

Answers to Problems; page 129:

1. (a) $2a$
   (b) $17p$
   (c) 0
   (d) $12a + 3c + 3c^2$
   (e) $6a + 4b + c$
   (f) $6p + 11q$
   (g) $3x^2 + (-x) + 1$

2. (a) $\{-4\}$
   (b) $\{\emptyset\}$
   (c) $\{-9\}$
   (d) $\{7\}$
   (e) $\{1\}$
   (f) $\{2\}$

Answers to Problems; page 130:

1. $\frac{3}{8}ab^2c^2d$
2. $200b^3c^2d$
3. $3a^3b$
4. $28abc$

Answers to Problems; page 131:

1. $16 + (-6b) + 14b^2$
2. $18xy + 6xz$
3. $-12b^3c^2 + (-21b^2c^3)$
4. $20b^3 + 70b^2 + (-40b)$
5. $(-p) + (-q) + (-r)$
6. $(-21a) + 35b$
7. $12x^2y + 18x^2y^2 + 24xy^2$
8. $(-x^2) + x$

Answers to Problems; page 132:

1. (a) $x^2 + 10x + 16$
   (b) $y^2 + (-8y) + 15$
   (c) $6a^2 + (-17a) + 10$
   (d) $a^2 + 4a + 4$
   (e) $x^2 + (-36)$
   (f) $y^2 + (-9)$

3. (a) $3a^2 + 5a + 2$
   (b) $4x^2 + 23x + 15$
   (c) $8 + 13n + 5n^2$
   (d) $6p^2q^2 + (-10pq) + (-56)$
   (e) $16 + (-14y) + y^2 + y^3$
   (f) $15y^2 + (-11xy) + (2x^2)$

Answers to Problems; page 134:

1. $\frac{1}{3}, 2, -\frac{1}{3}, -2, \frac{4}{3}, \frac{1}{7}, \frac{6}{5}, -\frac{7}{3}, -\frac{1}{7}, \frac{10}{3}, 100, -100, \frac{20}{9}, -\frac{5}{34}$

Answers to Problems; pages 135-136:

1. (a) True
   (b) True
   (c) False
   (d) True
   (e) True

2. (a) $\{\frac{1}{2}\}$
   (b) $\{\frac{6}{7}\}$
   (c) $\{\frac{3}{2}\}$
   (d) $\{1\}$
   (e) $\{\frac{2}{3}\}$
   (f) $\{\frac{3}{2}\}$
   (g) $\{35\}$
   (h) $\{-\frac{9}{2}\}$
   (i) $\{15\}$
   (j) $\{-\frac{9}{2}\}$
   (k) $\{0\}$

Answers to Problems; pages 139-140:

1. (a) $\{6\}$
   (b) $\{6\}$
   (c) $\{-3\}$
   (d) $\{0\}$
   (e) $\{-\frac{7}{3}\}$
   (f) $\{-3\}$
   (g) $\{0\}$
   (h) $\{1\}$

2. (a) $\{9\}$
   (b) {all real numbers}
   (c) $\{\emptyset\}$
   (d) $\{6\}$
   (e) $\{65\}$
   (f) $\{8\}$
   (g) $\{47\}$

Answers to Problems; pages 141-142:

1. (a) $\frac{1}{15}$
   (b) $\frac{1}{-8}$
   (c) $\frac{1}{\frac{1}{5}}$
   (d) $\frac{1}{-\frac{1}{6}}$
   (e) $\frac{1}{\frac{5}{3}}$
   (f) $\frac{1}{0.3}$
   (g) $\frac{1}{-\frac{3}{4}}$

Answers to Problems; pages 146-147:

1. (a) $\frac{1}{6ab}$
   (b) $\frac{1}{21yz}$
   (c) $\frac{1}{3a^3b}$
   (d) $\frac{1}{6m^3n^3}$
   (e) 1

Answers to Problems; page 148:

1. 0

2. 0

4. (a) $\{20, 100\}$
   (b) $\{-6, -9\}$
   (c) $\{0, 4\}$
   (d) $\{\frac{5}{3}, \frac{1}{2}\}$
   (e) $\{1, 2, 3\}$
   (f) $\{\frac{1}{2}, -\frac{3}{4}\}$
   (g) $\{6\}$
   (h) $\{2\}$

Answers to Problems; pages 152-153:

1. (a) $-\frac{3}{2} < -\frac{4}{3}$
   (b) $-|-7| = -|7|$

3. (a) True
   (b) True
   (c) True
   (d) True for any real number a
   (e) True
   (f) False
   (g) False
   (h) all positive numbers
   (i) False

4. (a) $0 < x$
   (b) $x < 0$

Answers to Problems; pages 154-155:

1. (a) True
   (b) True
   (c) True
   (d) True

Answers to Problems; pages 159-161:

1. (a) $b = 9$
   (b) $b = 17$
   (c) $b = \frac{1}{2}$
   (d) $b = \frac{5}{6}$
   (e) $b = 91$
   (f) $b = \frac{1}{39}$
   (g) $b = 1.68$
   (h) $b = \frac{161}{120}$

3. (a) True   (d) True

(b) False   (e) True

(c) True   (f) True

4. (a) $5 < 13, \quad 8 < 13$

(b) $(-3) < (-1), \quad (-1) < 2, \quad (-3) < 2$

(c) $5 + 2 = 7, \quad 5 = 7 + (-2)$

6. (a) False   (c) True

(b) True

8. For any real numbers a and b

$$|a-b| \leq |a| + |b|$$

9. (a) $\{5\}$ (d) $\{26\}$

(b) $\{125\}$ (e) $\{97\}$

(c) $\{4\}$ (f) $\{.9\}$

Answers to Problems; pages 165-166:

1. (a) $x > x$ (e) $x > \frac{3}{2}$

(b) $x < 1$ (f) $x < \frac{17}{6}$

(c) $x < -4$ (g) $x > 2$

(d) $x > \frac{4}{3}$

3. (a) At least 6 books.

(b) All integers between 24 and 40.

Answers to Problems; page 176:

1. (a)

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

(b)

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

(c)

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

(d)

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

·	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

2. a, b

3. For each non-zero element there exists a multiplicative inverse.

__Answers to Problems__; page 179:

1. $\frac{5}{4}$
2. -1.262
3. -3.01
4. 16
5. 23
6. -21
7. 13
8. Yes, yes, no
9. $\{750\}$
10. -13°
11. 14,777 feet

__Answers to Problems__; pages 182-183:

1. (a) $7y$
   (b) $\frac{3}{2}c$
   (c) $-2a^2 + 6a$
2. $-3x + 4y$
3. 1
4. $1 - x$
5. $a - 11b$
6. $4x^2 + x + 15$

__Answers to Problems__; page 184:

1. (a) $3a - 6b + 3c$
   (b) $-7x$
   (c) $a - 2$
   (d) $a - 2b + 5c$
   (e) $-3ab - ac + 3a$
2. (a) $\{1\}$
   (b) $\{2\}$
   (c) all y such that $y > -\frac{1}{2}$
   (d) all u such that $u > -\frac{2}{3}$
   (e) all a
3. (a) $x = 12$
   (b) 51
   (c) < 23 students

Answers to Problems; pages 186-188:

1. (a) 8
   (b) 8
   (c) -8
   (d) 8
   (e) -4
   (f) 4
   (g) 6
   (h) 6
   (i) -7
   (j) 7

2. (a) $5 - x$
   (b) $|5 - x|$
   (c) $x + 2$
   (d) $|x + 2|$
   (e) $-x + 1$
   (f) $|-1 + x|$
   (g) $x - 0 = x$
   (h) $|x - 0| = |x|$

3. (a) $=$
   (b) $>$
   (c) $>$
   (d) $>$
   (e) $>$
   (f) $>$
   (g) $=$
   (h) $>$

5. 5, 3

6. $3 < x < 5$

7. $x < 3$ or $x > 5$

8. $x > 3$ and $x < 5$

9. (a) $\{-2, 14\}$
   (b) $\{4\}$
   (c) $\{8, 12\}$
   (d) Real numbers $-3 < x < 3$
   (e) All real numbers
   (f) $\{-1, 1\}$
   (g) Real numbers $4 < y < 12$
   (h) $\{\emptyset\}$
   (i) $\{-22, -16\}$
   (j) $\{-14, 4\}$

10. $x = -3$ and $x = 3$
    $x > -3$ and $x < 3$
    $x \geq -3$ and $x \leq 3$
    $x < -3$ and $x > 3$
    $x \geq -3$ and $x \leq 3$

11. (a) (1) 66; (2) 33; (3) $7\frac{1}{3}$

(b) (1) 5; (2) 8; (3) $10\frac{1}{3}$

(c) (1) 63; (2) 30; (3) $4\frac{1}{3}$

(d) (1) 42; (2) 9; (3) $6\frac{2}{3}$

Answers to Problems; page 190:

1. 6
2. -6
3. -6
4. $\frac{20}{3}$
5. 4
6. $\frac{7}{16}$
7. 2

Answers to Problems; pages 191-193:

3. (a) -45

(b) -200

(c) $3\sqrt{5}$

(d) $-\frac{2}{3}$

(e) 12

(f) 0

(g) impossible

5. (a) $\{7\}$

(b) $\{-7\}$

(c) $\{-7\}$

(d) $\{7\}$

(e) $\{100\}$

(f) $\{75\}$

(g) $\{12\}$

(h) $\{0\}$

(i) $\{\frac{2}{3}\}$

6. (a) $\{-9\}$ (c) $\{5\}$
   (b) $\{16\}$

7. $-\frac{16}{3}$ 8. 9

9. Dick is 7 years old and John is 21 years old.

10. 22, 24

11. The sum of any two consecutive numbers from the set of odd integers between 0 and 42 would satisfy the condition of the problem.

12. $1\frac{1}{4}$ hours. 13. 9

14. 12 pennies, 16 dimes, 22 nickels. Total $2.82.

15. $25. 16. $n < 12$

17. 40 gallons of maple syrup and 120 gallons of corn syrup.

Answers to Problems; pages 196-197:

2. (a) $\frac{4}{3}$ (b) $-\frac{4}{3}$

3. (a) $-n$ (b) $\frac{1}{n}$

4. (a) $\frac{2}{3}$ (c) $-\frac{2}{3}$
   (b) $\frac{2}{3}$

5. (a) y (c) $x + 1$
   (b) y (d) 1

6. (a) 2 (c) -2

(b) -2

7. (a) $\frac{x + 2}{3}$ (c) $\frac{2x + 1}{3}$

(b) $\frac{2x - 3}{2y - 3}$ (d) $x + 2$

8. (a) $2 - a$ (c) -1

(b) $-a$

9. (a) $x - 1$ (c) $\frac{3}{4}(x - 1)$

(b) $\frac{x + 1}{4}$ (d) $x - 1$

Answers to Problems; pages 198-199:

1. $\frac{21}{16}$

2. (a) $\frac{6}{5}$ (b) $\frac{12}{7}$

3. (a) $-\frac{10}{9}$ (b) $\frac{10}{9}$

4. (a) $\frac{x^2}{12}$ (b) $\frac{3}{4}$

5. (a) 5 (c) 5

(b) $\frac{29}{6}$ (d) $\frac{11}{6}$

6. (a) $\frac{x + 2}{4}$ (b) $\frac{3}{4}(x + 2)$

7. (a) $\frac{(n + 3)(n + 2)}{6}$ (c) $\frac{2(n + 3)}{3(n + 2)}$

(b) 1

8. (a) y^2 (b) $2a$

9. yes, no

10. $f = 126$ 11. $x = 278$

Answers to Problems; pages 200-202:

1. (a) $\frac{11}{9}$ (c) $\frac{1}{9}$

   (b) $-\frac{1}{9}$

2. (a) $\frac{9}{a}$ (c) $\frac{4a + 5}{a^2}$

   (b) $\frac{13}{2a}$

3. (a) $\frac{3x}{4}$ (c) $-\frac{x}{4}$

   (b) $\frac{x^2}{8}$ (d) $\frac{x}{4}$

4. (a) $\frac{19a}{35}$ (c) $\frac{20a - 1}{35}$

   (b) $\frac{20 - a}{35}$

5. $\frac{3(x - 2)}{5}$

6. (a) $\{12\}$ (e) $\{20\}$

   (b) $\{2\}$ (f) $\{5, -1\}$

   (c) $\{2\}$ (g) $1 < x < 5$

   (d) $\{18\}$

7. 150 and 90

8. Numerator was increased by 5.

9. 5 and 2, $\frac{7}{10}$, $\frac{3}{10}$.

10. (a) $\frac{d}{7}$
    (b) $\frac{1}{8}, \frac{d}{8}$
    (c) $\frac{1}{7} + \frac{1}{8}, \frac{d}{7} + \frac{d}{8}$
    (d) $\frac{56}{15}$
    (e) $\frac{15}{56}$

11. .45, .662, .312

Answers to Problems; page 204:

1. $\frac{3}{2}$
2. 2a
3. $\frac{a^3}{3y^3}$
4. 1
5. $\frac{48}{5}$
6. $-\frac{1}{48}$
7. 7
8. $\frac{(x + 8)(x + 2)}{27}$
9. $\frac{y}{3}$

Answers to Problems; page 207:

1. yes
2. yes
3. yes
4. yes
5. yes
6. $5 \cdot 17$
7. $3 \cdot 17$
8. not factorable
9. $2 \cdot 61$
10. $3 \cdot 47$

Answers to Problems; pages 208-209:

1. All except 106, 306, 806, 118, 5618; The clue is in the last two digits of the number.
2. All except 16, 106, 601, 61, 5129, 32122; The clue is in the sum of the digits.
3. Yes, No.

5. (a) yes
   (b) no
   (c) no
   (d) yes
   (e) yes

Answers to Problems; page 212:

1. Refer to Sieve of Eratosthenes.

Answers to Problems; page 214:

1. $2 \cdot 7 \cdot 7$
2. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$
3. $2 \cdot 3 \cdot 43$
4. $5 \cdot 5 \cdot 5 \cdot 5$
5. $2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$
6. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
7. $2 \cdot 3 \cdot 3 \cdot 7$
8. $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$
9. $3 \cdot 5 \cdot 5 \cdot 11$
10. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$
11. $2 \cdot 3 \cdot 3 \cdot 61$

Answers to Problems; pages 217-218:

1. (a) $\frac{1}{10}$
   (b) $\frac{29}{36}$
   (c) $\frac{8}{39}$
   (d) $\frac{37x}{72}$
   (e) $\frac{49}{180}$
   (f) $\frac{k}{2 \cdot 2 \cdot 2 \cdot 7}$
   (g) $\frac{805x - 6}{840}$
2. (a) True
   (b) False
   (c) False

3. (a) $>$ (c) $=$

   (b) $<$ (d) $<$

4. $94 + 47 = 141$

5. 97 plus its largest prime factor 97 is 194 cents.

6. $200 + \frac{1}{12}$, $200 + \frac{1}{12}$, $\frac{1}{3}$, 800

Answers to Problems; pages 220-222:

1. (a) Impossible to obtain an even sum.

   (b) 15 and 10, or 30 and 5.

   (c) 150 and 1, 50 and 3, 75 and 2, or 25 and 6.

2. 6 and 3, 9 and 2.

3. (a) 18, 16 (d) 15, 22

   (b) 4, 243 (e) No solution

   (c) 216, 1 (f) 9, 30

4. $\ell = 25$ feet, $w = 9$ feet.

Answers to Problems; pages 223-224:

1. 2^6, $2^4 \cdot 5$, 2^7, 3^4, 7^2, 2^5, 3^3, $2^3 \cdot 7$, 3^5, 2^9, 5^4, $2^8 \cdot 3$

2. positive, real

Answers to Problems; page 225:

1. (a) m^{14}
(b) x^{12}
(c) $4x^4$
(d) 2^4x^4
(e) 3^7a^4
(f) 2^9a^{10}
(g) $x^{2a + a}$
(h) 3^6
(i) $3^4 \cdot 2^3$
(j) $2^7 \cdot 3^5 \cdot 5^3$
(k) $9k^2m^2t^2$

2. False
3. True
4. False
5. False
6. False
7. False
8. True
9. False
10. True
11. False
12. True

13. (a) $2^4 \cdot 3$
(b) $-8x^6$
(c) $a^3 + a^4 - 2a^5$

Answers to Problems; pages 227-228:

1. (a) 16
(b) $\frac{1}{8}$
(c) $\frac{1}{8}$
(d) $\frac{2}{9}$

2. (a) $\frac{x^4}{4}$
(b) 36^2
(c) 1
(d) $\frac{4}{a}$

3. (a) $\frac{1}{a^2c^3}$
(b) $a^6b^6c^5$
(c) $a^2b^3c + a^4b^3c^4$

4. (a) $\frac{1}{5x}$

(b) $\frac{5 + x}{25x^2}$

(c) $5x$

5. (a) $\frac{6}{x^4y^3}$

(b) $\frac{6x^3}{17x^4a^6}$

(c) $\frac{9x^2y^3}{4a^6b^6}$

6. False

7. False

8. True

9. True

10. True

11. The reciprocal of zero is not a number.

Answers to Problems; pages 231-233:

1. (a) 3^2

(b) $\frac{1}{3^3}$

(c) b^2

(d) $\frac{1}{10}$

(e) 10^3

(f) 1

(g) $\frac{b^2}{a^3}$

(h) $\frac{9y^3}{2x^3}$

(i) $\frac{1}{3t^2}$

2. (a) 1

(b) 10^7

(c) $.007$

(d) $\frac{4}{a^3b}$

(e) $\frac{1}{2y^4}$

(f) $\frac{1}{8y^4}$

(g) $\frac{3^4}{2^6}$

(h) 1

(i) $\frac{y^5}{2x^4}$

3. (a) 93
   (b) $9 \cdot 3$
   (c) yes

4. (a) 6
   (b) -2
   (c) -8
   (d) 14
   (e) 4
   (f) -3
   (g) 9
   (h) -2

Answers to Problems; pages 234-236:

1. (a) $9a^6$
   (b) $3a^6$
   (c) $27a^6$
   (d) $3a^9$

2. (a) $\frac{x}{3y^2}$
   (b) $\frac{5x}{3y^2}$
   (c) $\frac{1}{3y^2}$

3. (a) a
   (b) $-a$
   (c) a^2
   (d) $-3a^3$

4. y^2

5. $-\frac{1}{z^{15}}$

6. $\frac{7a^2}{20}$

7. (a) x^a
   (b) x^{3a}
   (c) x^{6a}

8. $\frac{15}{2a^2}$

9. (a) yes

(b) no

(c) yes

(d) no

(e) yes

(f) yes

(g) yes

10. (a) Not the same.

(b) Not the same.

11. (a) $\frac{bc + ac + ab}{abc}$

(b) $\frac{55b^2 + 91ab - 245a^2}{175a^2b^2}$

14. (a) -288

(b) 576

(c) 1

15. (a) $x^5 + x^4x^3 + 2x^2 + 1$

(b) $4x^2 - 12xy + 9y^2$

(c) $a^3 + 3a^2b + 3ab^2 + b^3$

Answers to Problems; page 239:

1. (a), (b), (e) are polynomials over the integers
2. (a) $2x^2 - 4x$
    (b) $x^2y - 2xy^2$
    (c) $t^2 + t - 6$
    (d) $-\frac{9}{4}x^3y^3z$
    (e) $u^2 - \frac{1}{4}$
    (f) $x^2 + 4x + 4$
    (g) $18t^2 - 15t - 88$
    (h) $y^2 + y - 2$
3. Yes
4. No

Answers to Problems; pages 241-242:

1. (a), (e)
2. (a) $a(a + 2b)$
    (b) $3(t - 2)$
    (c) $a(b + c)$
    (d) $3xz(x - y)$
    (e) $a(x - y)$
    (f) $6(p - 2q + 5)$
    (g) $2\big((z + 1) - 3zw\big)$
    (h) $a^2b^3(a + b - 1)$
    (i) No factoring possible
    (j) $ab(x - y)$
    (k) $6r^2s(x - y)$
    (l) $(u^2 + v^2)(x - y)$
    (m) $(x - y)(4x - y)$
    (n) $2^23^2a^2b^2c^2$
3. (a) 1, 1, 2
    (b) 2, 1, 3
    (c) 3, 2, 5
    (d) 0, 5, 5
    (e) 2, 4
    (f) The degree of the product is equal to the sum of the degrees of the factors.

Answers to Problems; page 244:

1. $3xz(2x - y)$
2. $3st(3 - u)$
3. $36(4x^2 - 65 + 5y)$
4. $\frac{3v}{5}(2u^2 - 3uv + 5v)$
5. $-xy^2(x^2 - 2x - 1)$
6. $\frac{1}{36}ab(6 + 10a - 21b)$
7. $s\sqrt{3}(1 + s\sqrt{2})$
8. No common factor
9. $(a + 3)(x - 1)$
10. $(x + 3)(x + 1)$
11. $(u + v)(x - y)$
12. $(a - b)(a + b)$
13. $(x + y)u$
14. 0
15. $(x + y)(3x - 5y + 1)$

Answers to Problems; page 247:

1. $(a + 3)(x + 2)$
2. $(x + y)(u + v)$
3. $(a + 1)(2b + a)$
4. $(3s + 5)(r - 1)$
5. $(x - 1)(5 + 3y)$
6. 0
7. $(a + c)(a - b)$
8. $(t + 3)(t - 4)$
9. Not factorable
10. $(2a - 3b)(a - b\sqrt{3})$
11. $(2 + u + v)(a - b)$
12. $(x + 1)(x + 3)$
13. $(a + b)(a - b)$

Answers to Problems; pages 249-252:

1. (a) $a^2 - 4$
   (b) $4x^2 - y^2$
   (c) $m^2n^2 - 1$
   (d) $9x^2y^2 - 4z^2$
   (e) $a^4 - b^4$
   (f) $x^2 - a^2$
   (g) $2x^2 + 3xy - 2y^2$
   (h) $r^3 + r^2s^2 - rs - s^3$

2. (a) $(1 - n)(1 + n)$
(b) $(5x - 3)(5x + 3)$
(c) $4(2x - y)(2x + y)$
(d) $(5a - bc)(5a + bc)$
(e) $5(2s - 1)(2s + 1)$
(f) $4x(2x - 1)(2x + 1)$
(g) $(7x^2 - 1)(7x^2 + 1)$
(h) $(x - 2)(x + 2)$
(i) Not factorable
(j) Not factorable
(k) $3(x - 1)(x + 1)$
(l) $(a - 2)a$
(m) $4mn$
(n) $(x - y)(x + y - 1)$

3. (a) $\{3, -3\}$
(b) $\{\frac{1}{3}, -\frac{1}{3}\}$
(c) $\{\frac{1}{5}, -\frac{1}{5}\}$
(d) $\{2, -2\}$
(e) $\{0, \frac{1}{2}, -\frac{1}{2}\}$
(f) $\emptyset$
(g) $\{2, -2\}$
(h) $\{1, -5\}$

4. (a) 396
(b) 1591
(c) $884r$
(d) $391xy$
(e) 9999
(f) $2000mn$
(g) $1584m^2 - 1584n^2$
(h) 1584

5. (a) No
(b) No
(c) Yes
(d) Yes

6. (a) $\frac{2}{23}(5 - \sqrt{2})$
(b) $\frac{7 + 3\sqrt{5}}{2}$
(c) $-\frac{3}{2}$
(d) $3\sqrt{2} + \sqrt{15}$

7. (b) $(t + 1)(t^2 - t + 1)$
(c) $(s + 2)(s - 2s + 4)$
(d) $(3x + 1)(9x^2 - 3x + 1)$

8. (b) $(t - 1)(t^2 + t + 1)$
(c) $(s - 2)(s^2 + 2s + 4)$
(d) $(2x - 1)(4x^2 + 2x + 1)$

Answers to Problems; page 255:

1. (a) 9
   (b) 16
   (c) 36
   (d) t^2
   (e) x^2
   (f) 100
   (g) 12s
   (h) 9

2. (a), (c), (d), (e), (g) and (h).

3. (a) $(a - 2)^2$
   (b) $(2x - 1)^2$
   (c) $(x - 2)(x + 2)$
   (d) Not factorable
   (e) $(2t + 3)^2$
   (f) $7(x + 1)^2$
   (g) Not factorable
   (h) $(2z - 5)^2$
   (i) Not factorable
   (j) $(3a - 4)(3a - 2)$

4. (a) $x^2 + 6x + 9$
   (b) $x^2 + 2\sqrt{2x} + 2$
   (c) $a^2 + 2ab + b^2$
   (d) $x^2 - 2xy + y^2$
   (e) $x^2 - 2x + 1 - a^2$
   (f) $5 + 2\sqrt{6}$
   (g) 10,201

Answers to Problems; page 257:

1. (a) $(x + 1)(x + 3)$
   (b) $(x - 4)(x - 2)$
   (c) $(x - 4)(x + 2)$

2. (a) 1
   (b) 8 or -8
   (c) None

3. (a) $\{-5\}$
   (b) $\{\frac{5}{2}\}$
   (c) $(3a + 1)^2$ is ≥ 0 for every a
   (d) $\{2\}$

Answers to Problems; pages 260-261:

1. (a) $(a + 5)(a + 3)$
   (b) $(a - 5)(a - 3)$
   (c) $(a + 5)(a - 3)$
   (d) $(a - 5)(a - 3)$

2. (a) $(t + 10)(t + 2)$
   (b) $(t + 20)(t + 1)$
   (c) $(t + 5)(t + 4)$
   (d) Not factorable

3. (a) $(a + 11)(a - 5)$
   (b) $(x - 3)(x - 2)$
   (c) $(u - 6)(u - 4)$
   (d) $(y - 18)(y + 1)$
   (e) Not factorable

4. (a) $(z - 2)(z^2 + 2z + 4)(z + 1)(z^2 - z + 1)$
   (b) $(b^2 - 7)(b - 2)(b + 2)$
   (c) $(a - 3)(a + 3)(a - 2)(a + 2)$
   (d) $(y - 3)(y + 3)(y^2 + 9)$

5. (a) $(a + 7)(a - 2)$
   (b) Not factorable
   (c) $(a - 12)(a - 9)$
   (d) $(a + 40)(a - 15)$

6. (a) $\{12, -3\}$
   (b) $\{3, 2\}$
   (c) $\{4, 9\}$
   (d) $\{0, -6\}$
   (e) $\{3, -2\}$
   (f) $\{-4, 3\}$

7. (a) $\{7, -1\}$
   (b) $\{-12, 7\}$
   (c) 1 or 9

8. length is 7 feet, width is 5 feet.

Answers to Problems; page 265:

1. (a) $(x + 1)(2x + 3)$
   (b) $(2x + 1)(x + 3)$
   (c) Not factorable
2. $(3a + 7)(a - 1)$
3. $(4y - 1)(y + 6)$
4. $(3x + 1)(x - 6)$
5. $(3x + 2)^2$
6. $3a(3a + 1)$
7. $(2x - 15)(5x + 3)$
8. $(6 + a)(1 - 4a)$
9. $(5x - 7y)^2$
10. (a) $6(x - 25)(x + 1)$
    (b) $(x - 6)(6x + 25)$
    (c) $6(x + 5)^2$
    (d) $(x - 6)(6x - 25)$
    (e) Not factorable over the integers
    (f) $(3x + 10)(2x + 15)$
    (g) $3(x - 2)(2x - 25)$
    (h) $3(x - 2)(2x + 25)$
11. No
12. $(3x - 4)(x + 3)$ Yes
13. 4 and 11
14. 5 and 7, or -3 and -1
15. width is 2 feet, length is 12 feet.

Answers to Problems; page 271:

1. $\frac{3}{x + 1}$
2. x^2
3. $\frac{x + 2}{x + 1}$
4. $\frac{b}{1 + b}$
5. $\frac{(x + 3)(x + 1)}{2x}$
6. 1

Answers to Problems; page 273:

1. $\frac{15 - 2x}{5x^2}$
2. $\frac{bc + ac + ab}{abc}$
3. $\frac{2 - a - 4a^2}{2a^2}$
4. $\frac{4 + x}{x - 1}$
5. $\frac{5m - 8}{(m - 1)(m - 2)}$
6. $-\frac{8x}{(x + 5)(x - 3)}$
7. $\frac{5m - n}{(m - n)n}$
8. $\frac{x^2 - 2xy - y^2}{(x - y)(x + y)}$
9. $\frac{9 - 5x}{3x(x + 2)}$
10. $\frac{6a - 10}{a(a + 7)(a + 5)}$
11. $x - y$
12. $\frac{ab}{b - a}$
13. Yes

Answers to Problems; page 277:

1. (a) $-12a^2 - 7a + 12$ (b) $-2x^3 - 2x^2 - 7x + 8$
2. (a) $13a - 20$ (b) $-11x^2 - 6x - 6$

Answers to Problems; page 278:

1. $2x + \frac{3}{x - 2}$
2. $2x - 5$
3. $2x^2 + 10x + 60 + \frac{365}{x - 6}$
4. $2x^4 + 2x^3 + 3x^2 - 2x - 2$
5. $x^2 - x + 5$

Answers to Problems; page 279

1. $x^2 + 7 + \frac{20}{x - 3}$
2. $x + 3 + \frac{30}{x - 5}$
3. $5x^2 - 10x + 9 + \frac{-11}{x + 2}$
4. $2x - 5$

5. $2x^2 + x - 1 + \frac{2}{3x - 2}$

6. $x^3 + x^2 + x + 1$

7. $N = QD + R$

Answers to Problems; page 281:

1. (a) $2x - 1 + \frac{10x + 2}{x^2 + 3}$
   (b) $x + \frac{8x - 1}{x^2 - 2x - 1}$
   (c) $3x^2 + 4x + 10 + \frac{16x + 33}{x^2 - 4}$

2. (a) $3x^5 - 5x^4 + 1$
   (b) $x^6 - x^3 + 1$
   (c) $2x^2 + 2x - 1$
   (d) $2x^4 + x - 5$ is not a factor

Answers to Problems; pages 286-288:

1. (a), (b), (c), (d), (g), (h), (i), (k), (m)

2. (a) $y = 12$
   (b) $x = 20$
   (c) $t = -1$
   (d) $s = \frac{1}{15}$
   (e) $x = 2$
   (f) $y = 1$

3. (a) $\{1\}$
   (b) $\{\frac{25}{6}\}$
   (c) $\{80\}$
   (d) $\{0\}$
   (e) $\{6\}$
   (f) $\{-5\}$
   (g) $\{\frac{3}{2}\}$
   (h) $\emptyset$
   (i) $\{0\}$

4. (a) Yes
   (b) Yes
   (c) No

Answers to Problems; pages 290-291:

1. (a) Yes, No
   (b) No, No
   (c) No, No
   (d) No, No
   (e) Yes, Yes
   (f) Yes, Yes
   (g) Yes, Yes
   (h) Yes, Yes
   (i) Yes, Yes
   (j) Yes, Yes
   (k) No, No
   (l) No, No

2. (a) $\{3\}$
   (b) $\{0\}$
   (c) $\{\frac{1}{3}\}$
   (d) $\emptyset$
   (e) All real numbers except -1
   (f) $\{2\}$

3. 6 inches wide, and 9 inches long

4. There are none

5. 6 or 2

6. 1200 girls

8. 10 and 15

Answers to Problems; page 293:

1. (a) $x < 27$
   (b) All real numbers $> \sqrt{2}$
   (c) All real numbers $< \sqrt{3}$
   (d) All real numbers < 12
   (e) All real numbers
   (f) $\emptyset$
   (g) All real numbers

2. (a) All real numbers between 0 and $\frac{1}{4}$
   (b) All real numbers between $\frac{1}{3}$ and 1
   (c) All real numbers between $-\frac{1}{2}$ and $\frac{1}{2}$
   (d) All real numbers which are either less than $-\frac{1}{2}$ or greater than $\frac{1}{2}$
   (e) All real numbers between -1 and 3
   (f) All real numbers between $-\frac{1}{2}$ and $\frac{1}{2}$

4. $y < -2$, all real numbers less than -2, $x > 1$, all real numbers > 1

5. All real numbers less than -1

Answers to Problems; pages 294-295:

1. (a) $\{-2, 5\}$
   (b) $\{-3, -1, 2, 0\}$
   (c) $\{\frac{1}{3}, -\frac{1}{2}, \frac{3}{4}\}$

2. (a) $\{2, -1\}$
   (b) $\{11, -11\}$
   (c) $\{1, -1, -3, -2\}$
   (d) $\{\sqrt{5}, -\sqrt{5}, 2\sqrt{6}, -2\sqrt{6}$
   (e) $\{0, 5, -5\}$
   (f) $\{-\frac{1}{2}, 3\}$
   (g) $\{0, 1\}$

3. $\{3, 1\}$

Answers to Problems; page 296:

1. (a) $\{0, 6\}$
   (b) $\{-3, 2, -2\}$
   (c) $\{2, -3\}$
   (d) $\{2, -2, 0\}$
   (e) $\{5, 3\}$

2. No, when $x = 1$

Answers to Problems; page 298:

1. $\{-\frac{1}{10}\}$
2. $\{60\}$
3. $\{1\}$
4. $\{-\frac{3}{30}\}$
5. $\{2\}$
6. $\{-\frac{1}{3}, -3\}$
7. All real numbers except 2
8. $\{0, 1\}$
9. $\emptyset$
10. $\{0\}$
11. (a) $1\frac{1}{5}$ hours
    (b) 6 hours
    (c) $\frac{1}{2}$ hour

Answers to Problems; page 299:

See Text.

Answers to Problems; pages 301-302:

1. $\emptyset$
2. $\{0\}$
3. $\{0, -1\}$
4. $\{9\}$
5. $\{3\}$
6. $\{-\frac{1}{3}, 1\}$
7. $\{1\}$
8. $\{5, -1\}$
9.
10. $\{\frac{1}{2}\}$
11. $S = 625$

Answers to Problems; pages 306-307:

1. All the points whose ordinates are -3 are on a line parallel to the horizontal axis and 3 units below it.

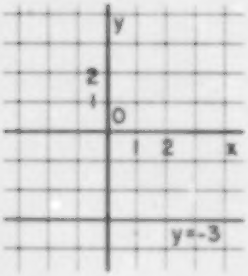


Figure for Problem 1.

2. Line (a) includes all possible points, such that each has its abscissa equal to the opposite of the ordinate. Line (b) includes those points, such that each has ordinate twice the abscissa.
Line (c) includes the points, such that each has ordinate that is the opposite of twice the abscissa.
All of these graphs are lines, and all pass through the origin. Their equations are:

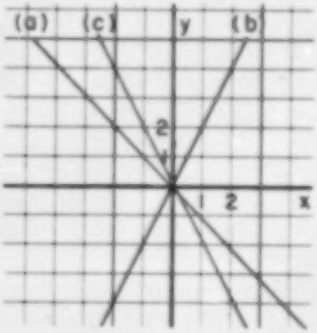


Figure for Problem 2.

(a) $y = -x$ (b) $y = 2x$ (c) $y = -2x$

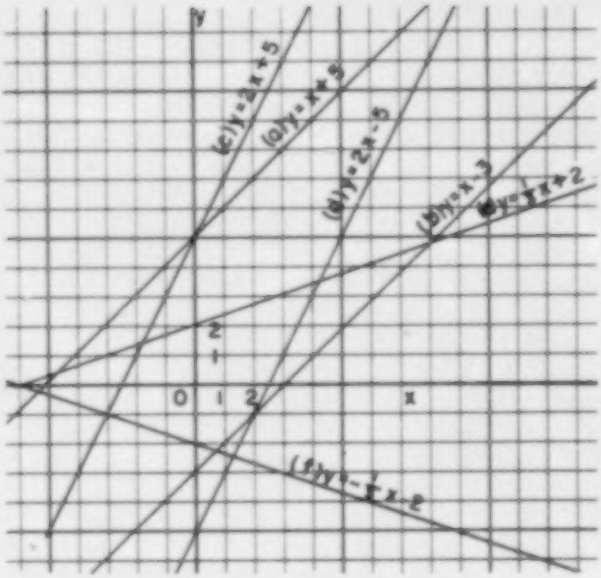


Figure for Problem 3.

3. The graph of (a) differs from the graph of (b) in the fact that it cuts the y-axis at a point 8 units above the point where the graph of (b) cuts it. The graph of (c) cuts the y-axis at a point 10 units above the point where the graph of (d) cuts it.
The graph of (e) not only cuts the y-axis at a different point than the point where the graph of (f) cuts it, but also the graph of (e) rises while the graph of (f) descends.
The graphs of (a) and (b) appear to be a pair of parallel lines. The graphs of (c) and (d) also appear to be parallel, but the graphs of (e) and (f) are not.

Answers to Problems; pages 309-311:

1. The open sentence whose truth set is the set of ordered pairs for which the ordinate is two greater than the abscissa is "$y = x + 2$". The graph of the set of points associated with this set of ordered pairs is the line shown in the figure.

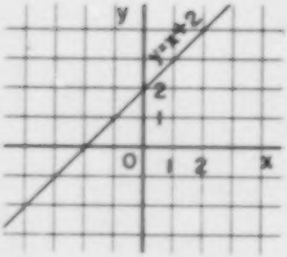


Figure for Problem 1.

It is not possible to draw the graphs of both of the sentences "$y > x + 2$" and "$y \geq x + 2$", because in the first one the line whose equation is "$y = x + 2$" is dotted, and in the second one it is a solid line.

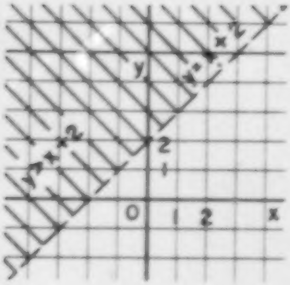


Figure for Problem 1(a).

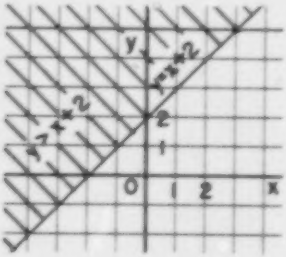


Figure for Problem 1(b).

2. In the sentence "$y = |x|$", since x is positive for all values of x, it follows that y is never negative. The solutions for which the abscissas are given are: (-3, 3), (-1, 1), $(1\frac{1}{2}, 1\frac{1}{2})$, (2, 2), (4, 4).

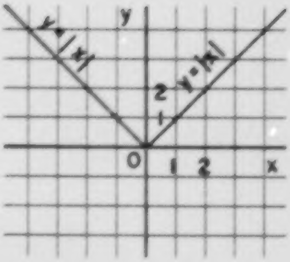


Figure for Problem 2.

3. (a) $y = \frac{3}{2}x$

x	-6	-4	0	4	8
y	-9	-6	0	6	12

(b) $y = \frac{3}{2}x - 3$

x	-4	-2	0	2	4
y	-9	-6	-3	0	3

(c) $y = \frac{3}{2}x - 6$

x	-2	-1	0	4	10
y	-9	$-7\frac{1}{2}$	-6	0	9

(d) $y = \frac{3}{2}x + 3$

x	-6	-2	0	4	6
y	-6	0	3	9	12

(e) $y = \frac{3}{2}x + 6$

x	-8	-6	0	1	2
y	-6	-3	6	$7\frac{1}{2}$	9

The graphs of all of these are lines parallel to each other.

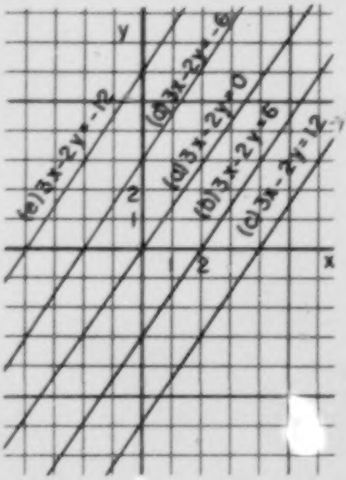


Figure for Problem 4.

4.

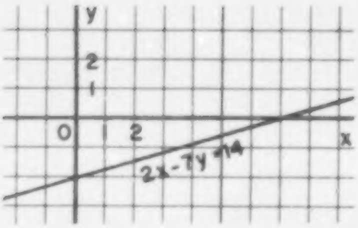


Figure for Problem 4(a).

(a) $2x - 7y = 14$

$$y = \frac{2}{7}x - 2$$

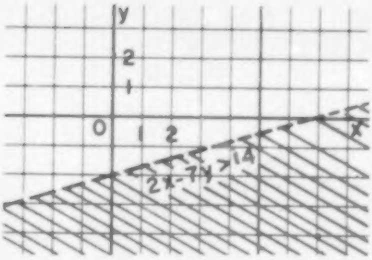


Figure for Problem 4(b).

(b) $2x - 7y > 14$

$$y < \frac{2}{7}x - 2$$

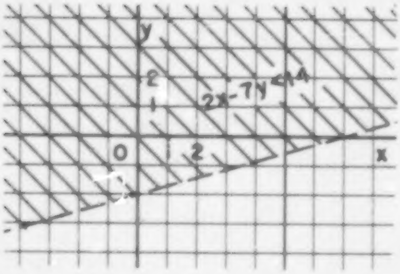


Figure for Problem 4(c).

(c) $2x - 7y < 14$

$$y > \frac{2}{7}x - 2$$

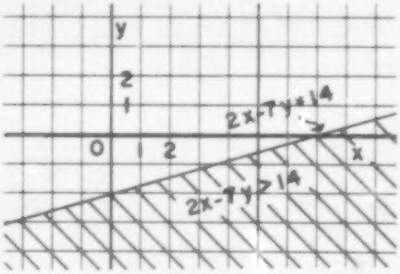


Figure for Problem 4(d).

(d) $2x - 7y \geq 14$

$$y \leq \frac{2}{7}x - 2$$

5. (a) $5x - 2y = 10$

$$y = \frac{5}{2}x - 5$$

(b) $2x + 5y = 10$

$$y = -\frac{2}{5}x + 2$$

(c) $5x + y = 10$

$$y = -5x + 10$$

(d) $3x - 4y = 6$

$$y = \frac{3}{4}x - \frac{3}{2}$$

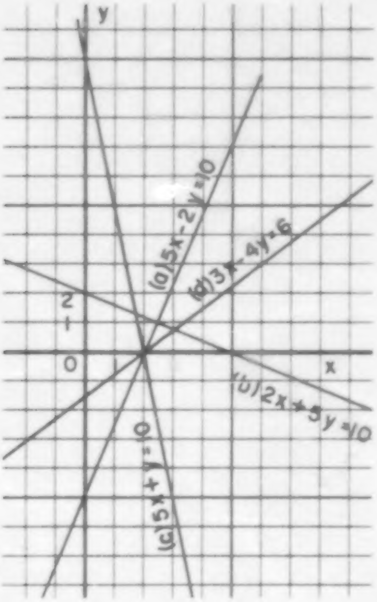


Figure for Problem 5.

7. (a)

x	-3	-2	$-\sqrt{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\sqrt{2}$	2	3
y	9	4	2	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	2	4	9

(b)

x	-3	-2	$-\sqrt{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\sqrt{2}$	2	3
y	-9	-4	-2	-1	$-\frac{1}{4}$	0	$-\frac{1}{4}$	-1	-2	-4	-9

(c)

x	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	$2\frac{1}{2}$	3
y	10	5	2	$1\frac{1}{4}$	1	$1\frac{1}{4}$	2	5	$7\frac{1}{4}$	10

(d)

x	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
y	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	-2	-4	no value	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$

Note that we approximate $\sqrt{2}$ by 1.4 when drawing graphs.

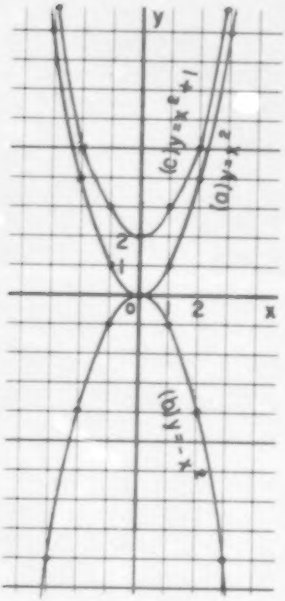


Figure for Problem 7, (a) - (c).

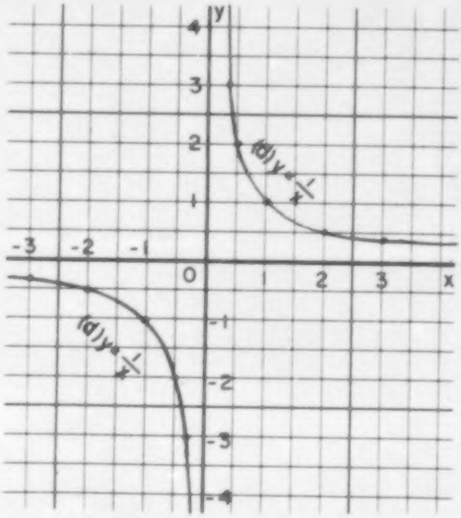


Figure for Problem 7(d).

Answers to Problems; page 315:

1. (a) $\frac{5}{13}$
   (b) 0
   (c) $\frac{7}{12}$
   (d) -2
   (e) $\frac{13}{5}$
   (f) no slope
   (g) $\frac{1}{3}$
   (h) $-\frac{4}{7}$

Answers to Problems; page 316:

1. $y = \frac{4}{3}x + 6$
2. $y = \frac{4}{3}x - 12$
3. $-\frac{4}{3}$
4. $-\frac{2}{3}$
5. $y = -\frac{5}{6}x - 3$
6. Slope is: $\frac{7}{2}$; equation is: "$y = \frac{7}{2}x - 3$"
7. Slope is: 1; equation appears to be "$y = x$", but $(5, 6)$ and $(-5, -4)$ are not on this line. There is no line satisfying these conditions.

Answers to Problems; pages 318-321:

1. (a) Since the y-intercept appears to be $(0, 2)$, on the graph, the equation of the line is:
   $y = \frac{5}{6}x + 2$
   (b) The line containing $(-6, -3)$ which has no slope has equation:
   $x = -6$.

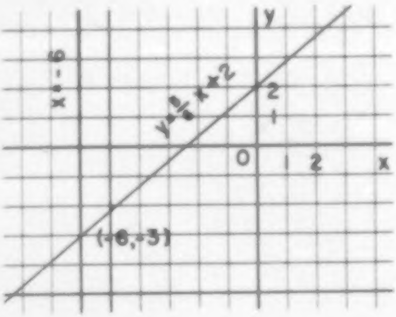


Figure for Problem 1.

2. Call attention to the distinction between the slopes of (c) and (e). For (c) the slope is 0 and the equation is "$y = 4$". For (e) there is no slope, since the denominator of the fraction form of the slope is $3 - 3$ or 0.

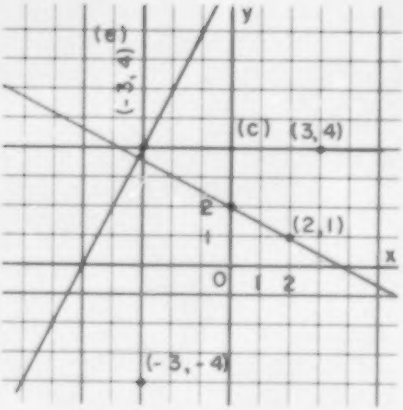


Figure for Problem 2.

3. The slope of the line containing (1, -1) and (3, 3) is

$$\frac{3 - (-1)}{3 - 1} \text{ or } 2.$$

The slope of the line containing (1, -1) and (-3, -4) is

$$\frac{-9 - (-1)}{-3 - 1} \text{ or } 2.$$

Hence, the point (-3, -9) is on the line containing (1, -1) and (3, 3).

4. (a) All of the lines have the same slope, $\frac{1}{2}$.

   (b) All of the lines have the same y-intercept number, -3.

   (c) The lines have the same slope, $-\frac{1}{2}$. Moreover, the two whose open sentences are "$\frac{1}{2}x + y = 3$" and "$2x + 4y = 12$" have the same y-intercept number, hence, they are the same line, and are parallel to the first one.

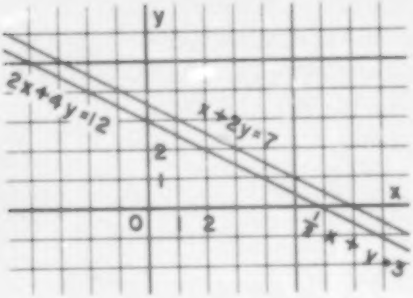


Figure for Problem 4(c).

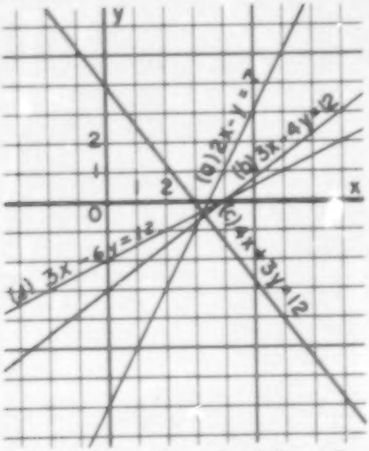


Figure for Problem 5.

5. (a) $y = 2x - 7$

(b) $y = \frac{3}{4}x - 3$

(c) $y = -\frac{4}{3}x + 4$

(d) $y = \frac{1}{2}x - 2$

The graphs of these are lines, because each open sentence is of the form
$Ax + By + C = 0$

6. (a) $y = \frac{2}{3}x$ (c) $y = -7x - 5$

(b) $y = -2x + \frac{4}{3}$ (d) $y = mx + b$

7. $y = \frac{1}{6}x + 7$

8. $y = -x - 1.$

Since -1 and $\frac{y - (-4)}{x - 3}$ are names for the same number

$$\frac{y - (-4)}{x - 3} = -1, \text{ provided } x \neq 3.$$

Then $y + 4 = (-1)(x - 3),$

$y = -x - 1.$

9. (a) $y = \frac{12}{5}x - 4$ (c) $y - 3 = 0$

(b) $y = -\frac{8}{5}x + 6$ (d) $y - 2 = \frac{1}{7}(x - 4)$

10\.

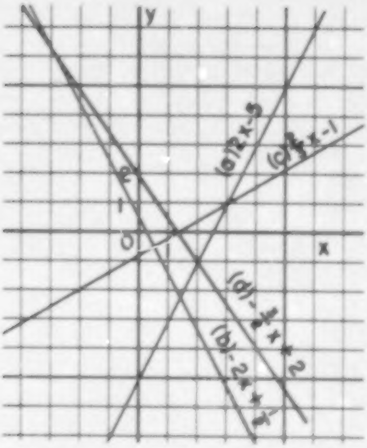


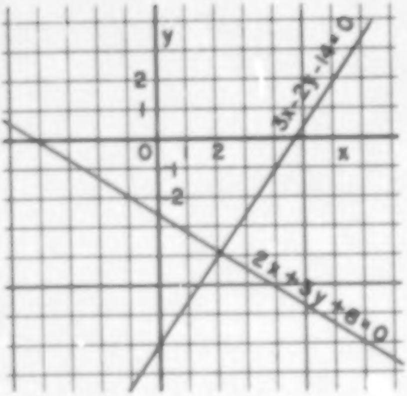
Figure for Problem 10.

11. (a) πd. This expression is linear in d. If the diameter is doubled, the circumference is doubled; if the diameter is halved, the circumference is halved. The ratio $\frac{c}{d}$ is equal to π; this ratio does not change when d is changed.

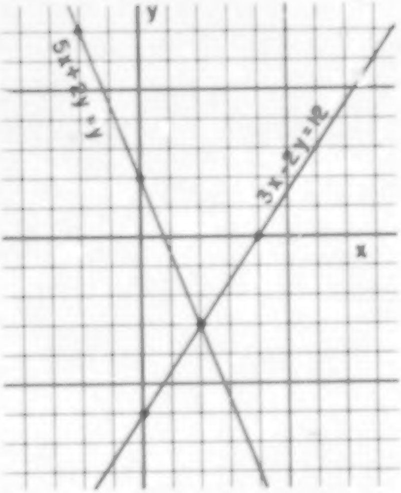
(b) $\frac{1}{4}\pi d^2$. (If the reader is not familiar with this, develop it as a combination of the two familiar relations, "area is πr^2" and "d is $2r$", or "r is $\frac{1}{2}d$"). This expression is not linear in d, but it is linear in d^2. If A is the area, $\frac{A}{d} = \frac{1}{4}\pi d$; $\frac{A}{d^2} = \frac{1}{4}\pi$. The value of $\frac{A}{d}$ changes when the value of d is changed; the value of $\frac{A}{d^2}$ does not change when d is changed.

Answers to Problems; pages 330-332:

1. (a) {2, -4}



(b) {(2, -3)}



(c) $\{(17, 12)\}$

(d) $\{(\frac{23}{2}, \frac{37}{2})\}$

(e) $\{(\frac{3}{5}, -\frac{9}{5})\}$

(f) $\{(\frac{6}{5}, \frac{7}{5})\}$

(g) $\{(15, 16)\}$

2. (a) $\{(7, -2)\}$

(b) $\{(\frac{1}{2}, 1)\}$

(c) $\{(\frac{7}{10}, \frac{4}{5})\}$

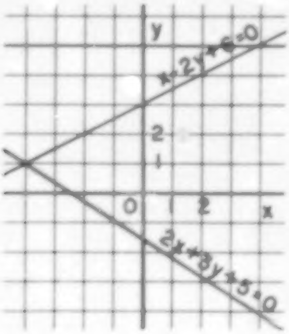
(d) No solution, why?

3. (a) 249 pupils, 62 adults.

(b) 141 three cent and 211 four cent stamps.

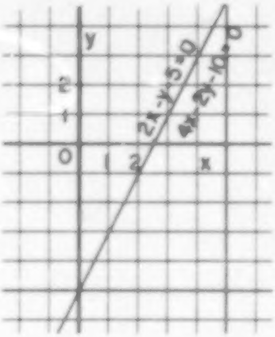
(c) No.

4. (a)



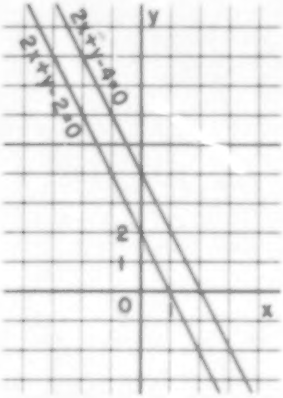
Truth set: $\{(-4, 1)\}$

(b)



Truth set: The whole line.

(c)



Truth set is $\emptyset$:
The lines are parallel.

5. $34x - 53y = 0$

Answers to Problems; pages 336-340:

1. (a) $\{(7, 2)\}$
   (b) Set of all coordinates of points on the line.
   (c) $\{(\frac{1}{2}, \frac{1}{4})\}$
   (d) $\emptyset$
   (e) $\{(\frac{28}{5}, 3)\}$
2. (a) $(-4, -6)$
   (b) $(12, 10)$
   (c) Set of all coordinates of points on the line.
   (d) $(3, -5)$
   (e) $(\frac{1}{2}, \frac{3}{2})$
   (f) $(-2, \frac{4}{3})$
3. (a) They intersect in the point.
   (b) Graphs coincide, Yes.
   (c) The graphs are parallel.
4. $\{(\frac{3}{2}, \frac{5}{2})$
5. (a) $\{(3, -4)\}$
   (b) $\{(0, 0)\}$
   (c) $\{(\frac{1}{2}, \frac{1}{3})\}$
   (d) $\{(1, \frac{1}{2})\}$
   (e) $\{(6, 1)\}$
   (f) $\emptyset$
   (g) $\{(\frac{7}{3}, \frac{8}{9})\}$
   (h) $\{(5, 7)\}$
6. 37, 19
7. 17, 13
8. 80 pounds of almonds, 120 pounds of cashews.

9. 37

10. 114 and 95 pounds.

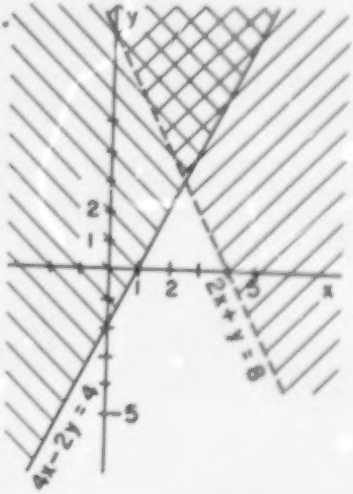
11. Apples 12 cents per pound and bananas are 18 cents per pound.

12. A walks at $3\frac{1}{4}$ m.p.h. and B walks at $2\frac{3}{4}$ m.p.h.

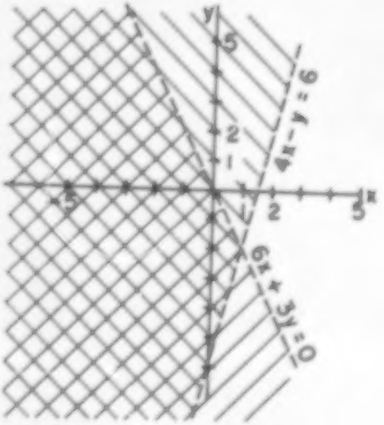
13. A's average was 60 m.p.h., B's average was 50 m.p.h.

Answers to Problems; page 342:

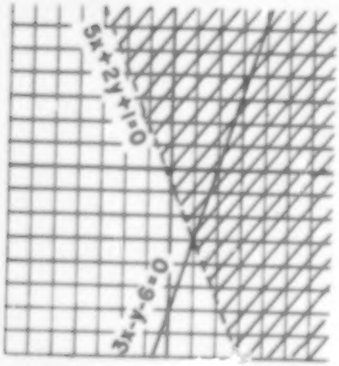
1.



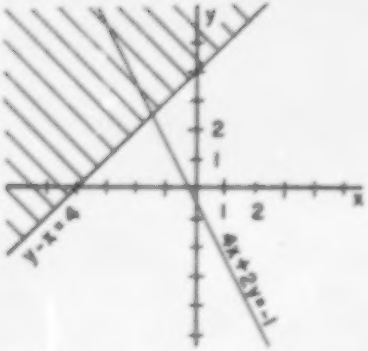
2.



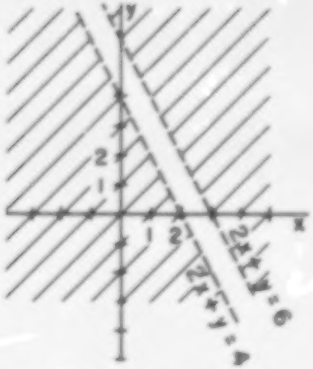
3.



4.

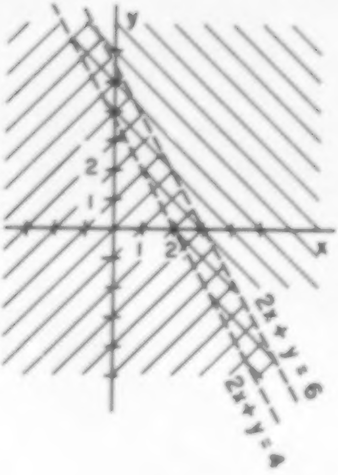


5.

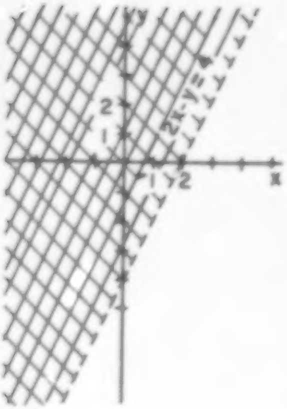


The empty set, $\emptyset$.

6.



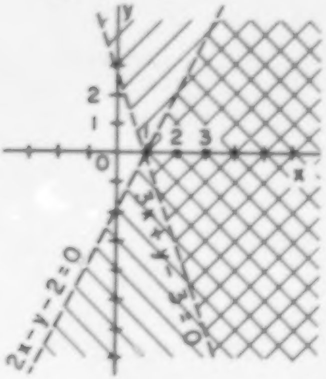
7.



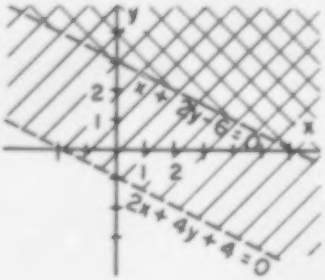
Answers to Problems; page 343:

The graphs of the truth sets here consist of all points in the unshaded and the doubly shaded regions of the figures.

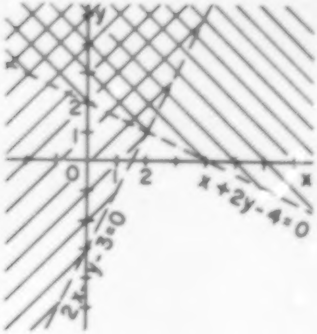
1. (a)



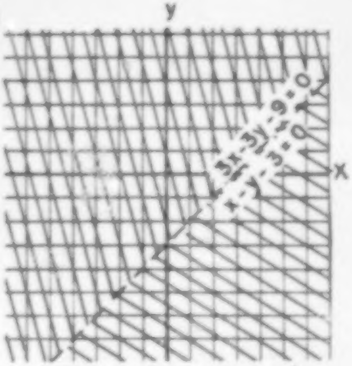
(c)



(b)

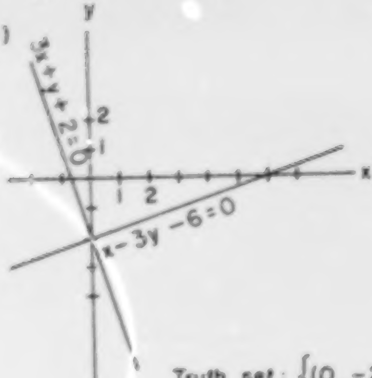


(d)



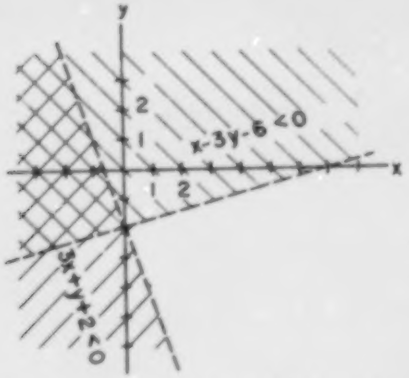
The null set, $\emptyset$.

2(a)



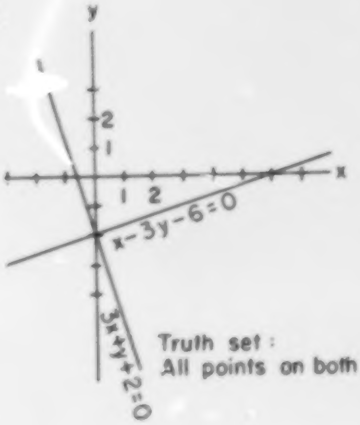
Truth set: $\{(0,-2)\}$

2(d)



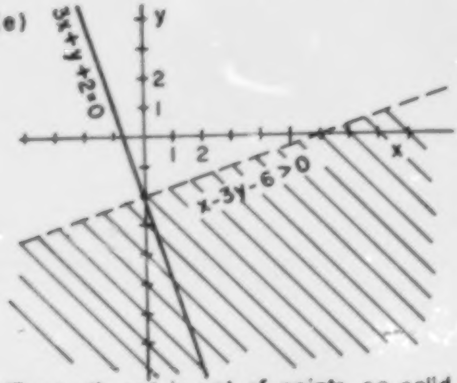
The truth set is doubly shaded region

2(b)



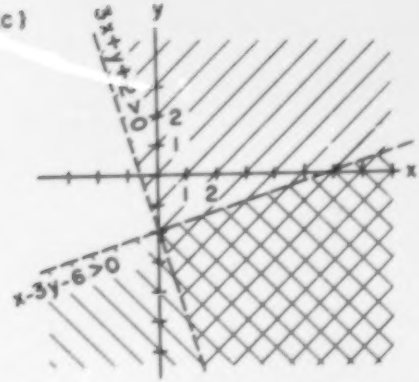
Truth set:
All points on both lines.

2(e)



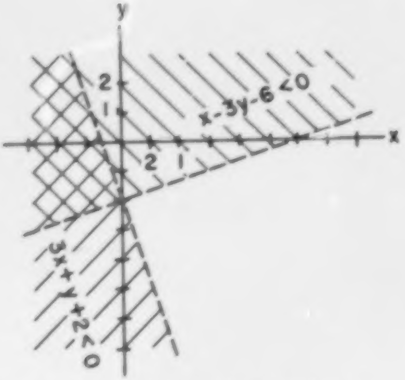
The truth set is set of points on solid line within shaded region

2(c)



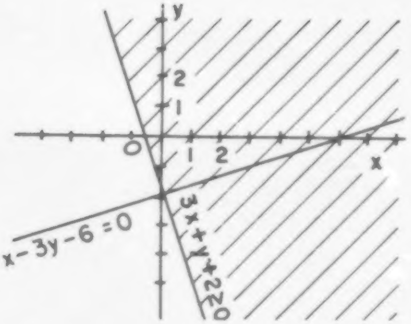
The truth set is doubly shaded region.

2 (f)



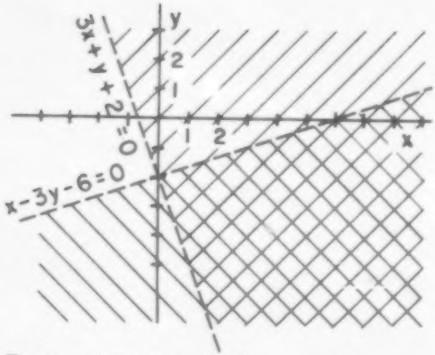
The truth set is whole shaded region.

2 (g)


y
2
1
0
1
2
x
x-3y-6=0
3x+y+2=0

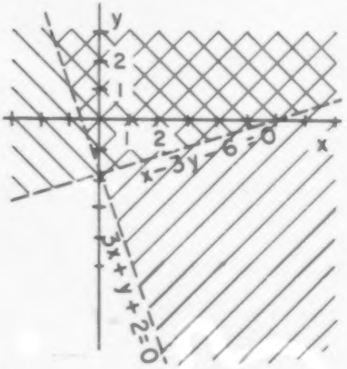
Truth set is whole shaded area and both lines

2(h)


y
2
1
1
2
x
3x+y+2=0
x-3y-6=0

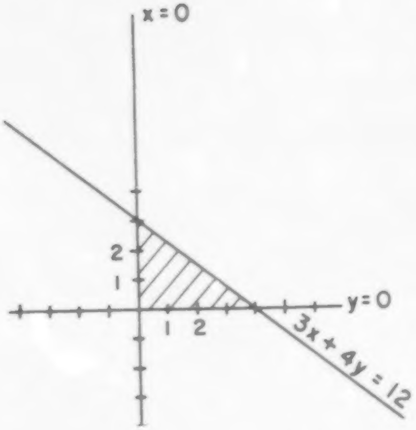
Truth set is the doubly shaded and the unshaded region.

2 (i)

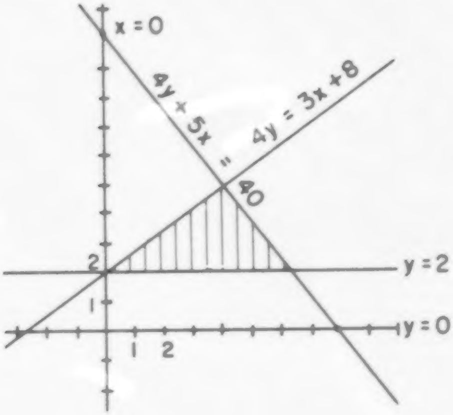

y
2
1
1
2
x
x-3y-6=0
3x+y+2=0

Truth set is the doubly shaded and the unshaded regions

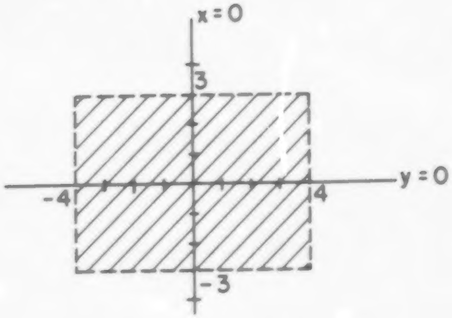
3(a)


x = 0
2
1
1 2
y = 0
3x + 4y = 12

3(b)


x = 0
4y + 5x = 40
4y = 3x + 8
2
1
1 2
y = 2
y = 0

3(c)



4. If r is the number of running plays and p is the number of pass plays, then $3r$ is the number of yards made on r running plays and $20(\frac{1}{3})$ p is the number of yards made on p passing plays. Since, the team is 60 yards from the goal line,

$$3r + \frac{20}{3}p \geq 60$$

if they are to score.

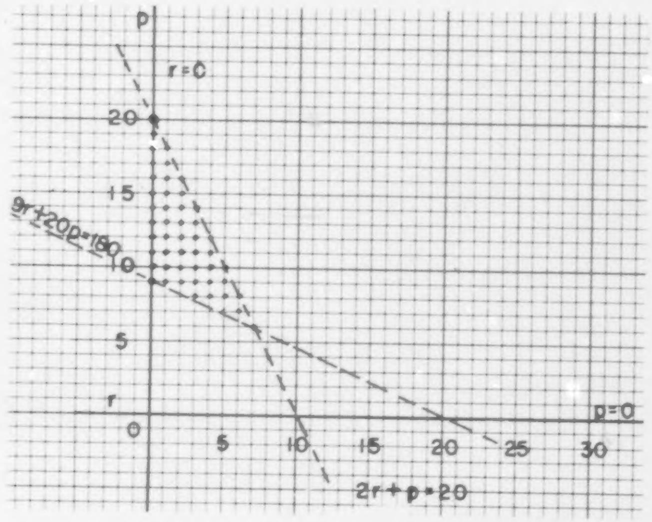
$30r$ seconds are required for r running plays, and $15p$ seconds are required for p passing plays; therefore,

$$30r + 15p \leq 5(60).$$

These two inequalities give the equivalent system

$$\begin{cases} 20p + 9r \geq 180 \\ p + 2r \leq 20 \end{cases}$$ (p and r are non-negative integers.)

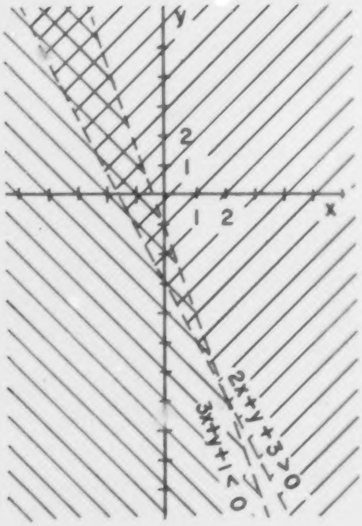
The graph of this system is.



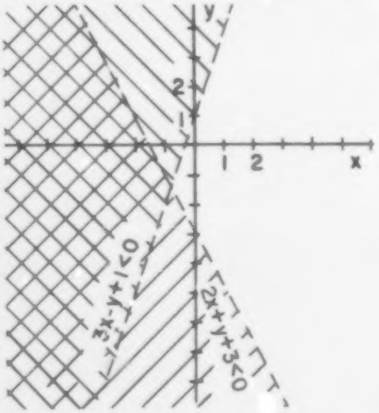
It is evident there are 48 different combinations of r and p which will assure success, for example, 2 running and 10 passing, etc. However, there are some combinations which leave a smaller time remaining, thus, giving the opponents less time to try to score. These are the points of the graph nearest the line $p + 2r = 20$.

Answers to Problems; pages 345-346:

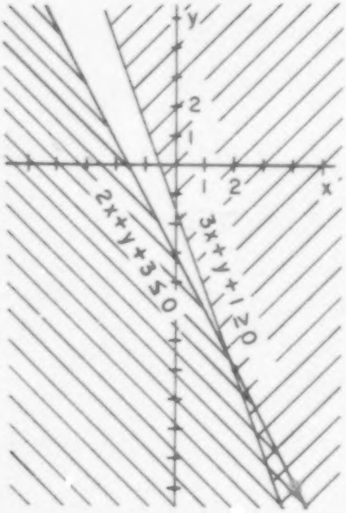
1.



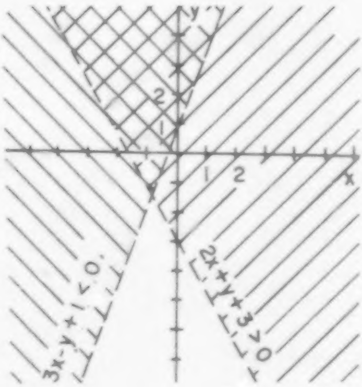
2.



3.


y
2
1
1
2
x
2x+y+3≤0
3x+y+1≥0

4.


y
2
1
1
2
x
3x-y+1<0
2x+y+3>0
